LABELED BINARY TREES AND SCHUR-POSITIVITY

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INTRODUCTION

In the 90's, Gessel introduced the following generating function,

$$B := \sum_{n\geq 1} \sum_{\substack{T\in\mathcal{T}_n^\ell \\ T \text{ standard}}} \mathrm{wt}(T)\frac{x^n}{n!} = \sum_{n\geq 1} B_n(a_1,a_2,b_1,b_2)\frac{x^n}{n!},$$

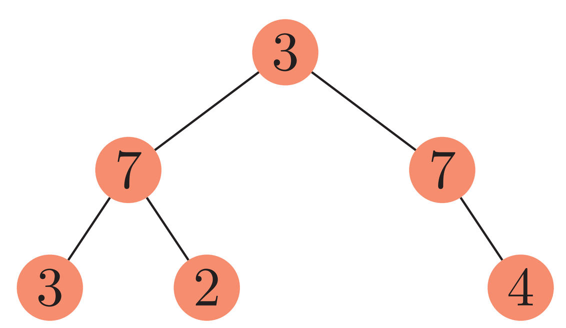
summing over standard labeled binary trees, and its multivariate generalization,

$$G := \sum_{n\geq 1}\sum_{T\in\mathcal{T}_n^\ell} \mathrm{wt}(T)x_T = \sum_{n\geq 1} G_n,$$

summing over all labeled binary trees, where $\mathrm{wt}(T) = a_1^{\mathrm{rasc}(T)}a_2^{\mathrm{rdes}(T)}b_1^{\mathrm{lasc}(T)}b_2^{\mathrm{ldes}(T)}$. He conjectured that G is Schur-positive. We prove this conjecture in Theorem 2 by expanding G in terms of ribbon Schur functions. We further show that a refinement of Schur-positivity to fixed canopy holds in Theorem 3. We apply our results to hyperplane arrangements and local binary search trees in Theorems 4, 5 and 6.

EXAMPLE AND EXPANSION

If T is the tree



then $\mathrm{wt}(T) = a_1a_2^2b_1b_2$ and $x_T = x_2x_3^2x_4x_7^2$.

Let B_n be the coefficient of $x^n/n!$ in B, and let G_n be the degree n component of G.

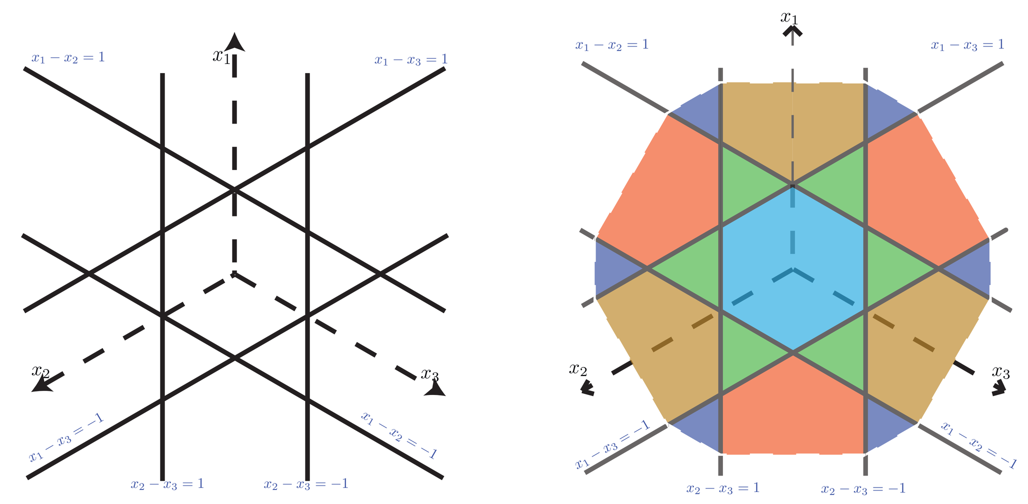
$$\begin{aligned} B_3 &= 1((a_1+b_2)^2 + a_1b_2) \\ &\quad + 2((a_1+b_2)(a_2+b_1) + a_1b_2) \\ &\quad + 2((a_1+b_2)(a_2+b_1) + a_2b_1) \\ &\quad + 1((a_2+b_1)^2 + a_2b_1). \\ G_3 &= r_{(3)}((a_1+b_2)^2 + a_1b_2) \\ &\quad + r_{(2,1)}((a_1+b_2)(a_2+b_1) + a_1b_2) \\ &\quad + r_{(1,2)}((a_1+b_2)(a_2+b_1) + a_2b_1) \\ &\quad + r_{(1,1,1)}((a_2+b_1)^2 + a_2b_1). \end{aligned}$$

HYPERPLANE ARRANGEMENTS

Specializations of B_n yield the number of regions in deformations of the braid arrangement.

$$\begin{aligned} B_n(1,1,1,1) &= n!\mathrm{Cat}(n) \\ &= \#\text{regions in Catalan arr.} \\ B_n(1,1,1,0) &= (n+1)^{n-1} \\ &= \#\text{regions in Shi arr.} \\ B_n(1,0,0,1) &= \#\text{regions in Linial arr.} \\ B_n(1,\zeta_6,\zeta_6^{-1},1) &= \#\text{regions in Semiorder arr.} \end{aligned}$$

Theorem 4 *The specialization $G_n(1,\zeta_6,\zeta_6^{-1},1)$ is the Frobenius characteristic of the symmetric group action on the regions of the semiorder arrangement given by permuting coordinates.*



Theorem 5 *The specialization $G_n(a_1,0,0,1)$ is the Frobenius characteristic of the S_n-action on Bernardi trees [3], graded by number of right children.*

SCHUR POSITIVITY

Let $H(z) = \sum_{n\geq 0} h_nz^n$. In unpublished work [1], Gessel proved that G satisfies the functional equation,

Theorem 1 (Gessel)

$$\frac{(1+a_1G)(1+b_2G)}{(1+a_2G)(1+b_1G)} = H((a_1b_2-a_2b_1)G + (a_1-a_2-b_1+b_2)),$$

thus proving G is symmetric in the x_i variables, and he conjectured that it is Schur-positive.

Theorem 2

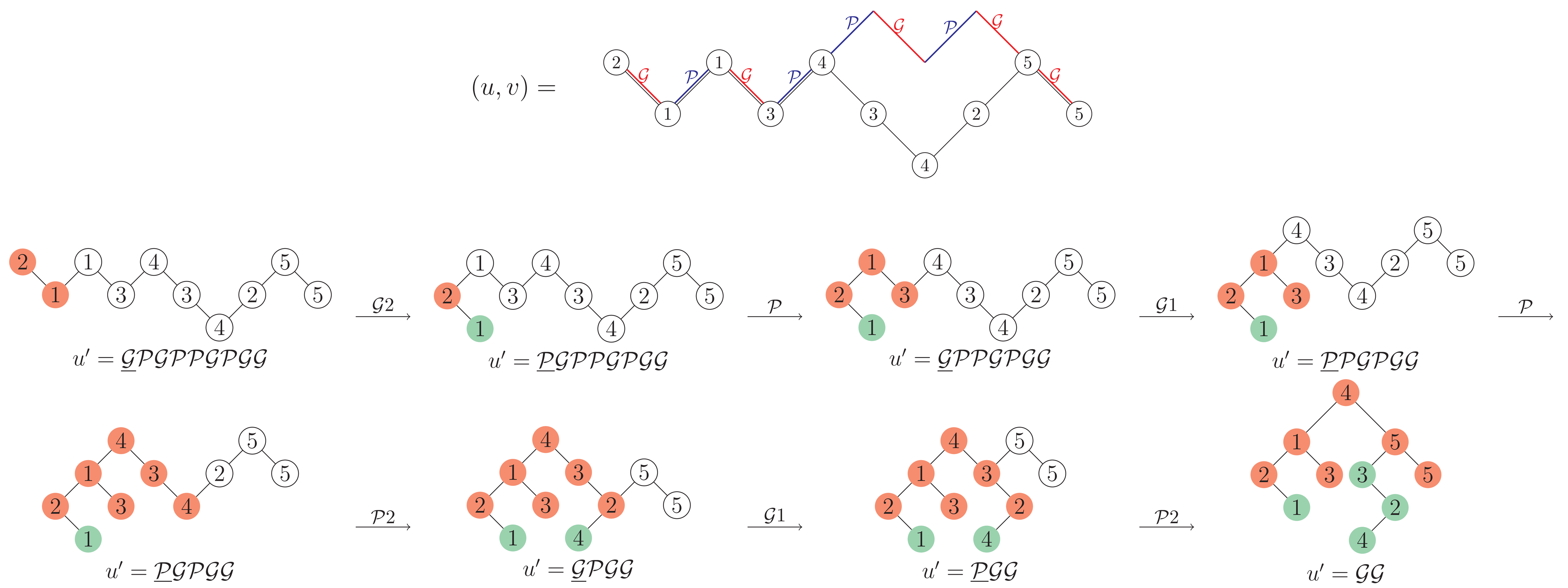
$$G = \sum_{n\geq 1}\sum_{\alpha\vDash n} r_\alpha(a_1b_2G + a_1 + b_2)^{n-\ell(\alpha)}(a_2b_1G + a_2 + b_1)^{\ell(\alpha)-1}.$$

Corollary 1 *G is Schur-positive. In particular, G expands positively in terms of ribbon Schur functions with coefficients in $\mathbb{N}[a_1+b_2, a_2+b_1, a_1b_2, a_2b_1]$.*

FIXED CANOPY

Theorem 3 *For any fixed canopy v of length $n-1$, let $G_{n,v}$ be the weighted sum over labeled trees on n nodes with canopy v. Then $G_{n,v}$ is Schur-positive.*

In order to prove Theorems 2 and 3, we define a weighted extension of the Push-Glide algorithm of Préville-Ratelle and Viennot [2] to relate labeled paths and labeled trees while keeping track of weights.



γ-POSITIVITY

Let $f_{n,k}$ be the number of standard local binary search trees on $[n]$ with $k-1$ right descents.

Theorem 6 *For $n\geq 1$, the distribution of right descents over the set of standard local binary search trees on n nodes is γ-nonnegative. As a corollary, the sequence of coefficients $\{f_{n,k}\}_{k\geq 1}$ is unimodal.*

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Ramsey Theory

Budapest Summer in Mathematics

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This summer, I spent eight weeks studying mathematics in the Budapest Semester in Mathematics program. Hungary is highly regarded for specializing in combinatorics, the branch of mathematics that I intend to pursue in graduate school; I was greatly appreciative of my opportunity to study in an environment focused and prominent in advancing the field. I took classes in combinatorics and number theory, and audited the "infamous" Conjecture and Proof problem solving course. In Number Theory, I was able to apply my background in abstract algebra and field theory developed at Santa Clara to the study of integers, modular arithmetic, and the exploration of prime numbers. In Advanced Combinatorics, our combinatorial structure of focus was hypergraphs, and many of the proofs we crafted included arguments using extremal theory, probabilistic method, linear algebra, and (almost daily) the pigeonhole principle. Having taken combinatorics and graph theory at Santa Clara in preparation for this course, I was excited to extend my knowledge of discrete mathematics as I set myself up to continue studying and researching in these fields.

Introduction to Ramsey Theory

Ramsey theory studies combinatorial structures and is interested in the size needed to ensure that an substructure with a desired property is contained within the disorganized structure. Typically applied to graphs and hypergraphs, Ramsey theory can be extended to the general philosophy

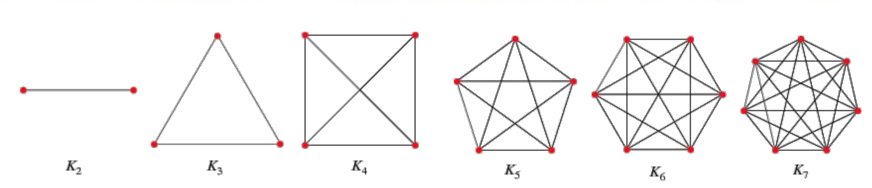
"There is no total chaos."

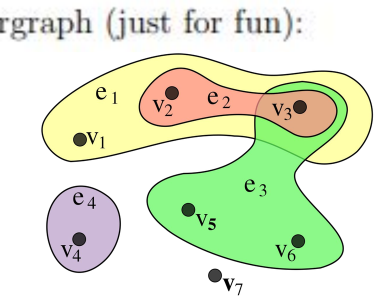
In other words, given a large enough disorganized system, it is certain that it will contain an organized subsystem.

Motivating Question: How many people do you need to ensure that among them there exist three who are all friends with one another or three who are all strangers with one another?

Graph Theory Background

Graph : A graph, G, is a set of vertices, a set of edges, and a relation assigning two vertices to each edge.
Subgraph : : A subgraph of a graph G is a graph contained in G (subset of vertex set, subset of edge set, same relation connecting vertices by edges as in G).
Complete Graph : A complete graph, K_n, is a graph on n vertices with the property that each pair of vertices is connected by an edge. *Note, K_n has $\binom{n}{2}$ edges.*
2 – coloring : An arbitrary assignment of two colors, usually red and blue, to a collection of items (e.g. vertices or edges of a graph)
Monochromatic $\mathbf{K_n}$: A complete graph on n vertices which has all edges in the same color.
Hypergraph : A hypergraph, $H = (V, E)$, consists of a vertex set V and an edge set E, where E is a collection of subsets of V.
Below, examples of monochromatic complete graphs, and a hypergraph (just for fun):





Definition of Ramsey Numbers and Theorem of Existence

Ramsey Number : Let $n_1, n_2 \in \mathbb{N}$. Define $R(n_1, n_2) = m$ to be the smallest integer such that given an arbitrary 2-coloring of the edges K_m, there exists a monochromatic complete subgraph of K_m on n_1 or n_2 vertices (i.e. monochromatic K_{n_1} or K_{n_2} contained in any random 2-coloring of K_m). *Note, we often denote $R(n, n)$ as $R(n)$.*

Ramsey Existence Theorem (for Graphs) : Assume that $n_1, n_2 \in \mathbb{N}$. There exists an integer $R(n_1, n_2) = m$ such that if the edges of K_m are colored red and blue any fashion, then there exists a monochromatic K_{n_1} or K_{n_2}.

Ramsey Existence Theorem (for Hypergraphs) : Assume that $n, t \in \mathbb{Z}$ satisfying $n \leq t$. There exists an integer m (depending on n, t) with the following property: if the edges of K^t_m are arbitrarily 2-colored, then there exists a monochromatic K^t_n (i.e. n vertices with all the $\binom{n}{t}$ edges determined by them having the same color).

Some Ramsey numbers are known, some are bounded, and most are beyond human capacity to calculate - regardless, they exist! Paul Erdős, a renowned and incredibly prolific Hungarian mathematician, once commented on finding R(5) and R(6):

"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

Pigeonhole Principle

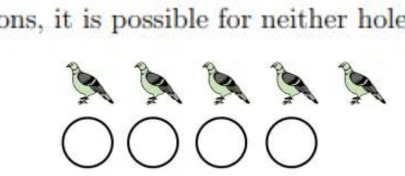
Preliminary Idea : Given n pigeons and $n-1$ pigeonholes, there exists at least one hole with at least two pigeons.
Generalized : Given N pigeons and k pigeonholes, there exists at least one hole with at least $\lceil \frac{N}{k} \rceil$ pigeons.
Ramsey Case : Given $2n-1$ pigeons and 2 pigeonholes, one hole will have at least n pigeons.
Proof : With $2n-1$ pigeons and 2 holes, we can find a hole with at least the following number of pigeons:

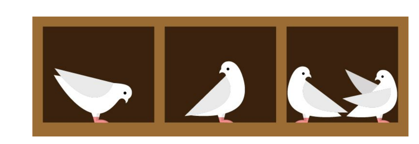
$$\lceil \frac{2n-1}{2} \rceil = \lceil n - \frac{1}{2} \rceil = n$$

Considering only $2n-2$ pigeons, we are not guaranteed a hole with n pigeons, since

$$\lceil \frac{2n-2}{2} \rceil = \lceil n-1 \rceil = n-1$$

Thus, with only $2n-2$ pigeons, it is possible for neither hole to hold n pigeons.

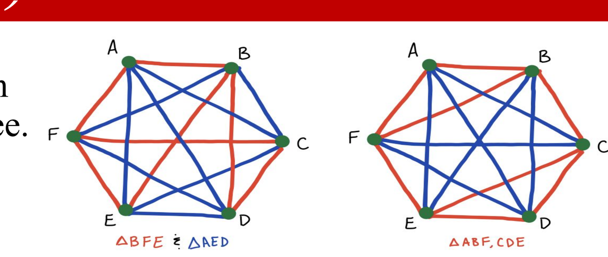




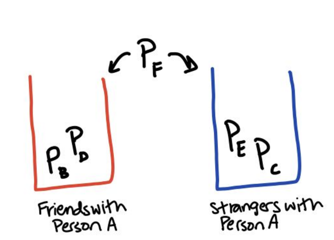
The idea of stuffing pigeons into holes sounds trivial, but PHP is a powerful counting argument implemented in profound proofs spanning a variety of branches of mathematics - set theory, number theory, analysis, and others (see supplemental materials for problems and proofs applying PHP). Ramsey's work was an extension of the pigeonhole principle, and thus PHP has a natural application to studying Ramsey numbers.

Proving R(3) = 6

Goal: Show that given any arbitrary 2-coloring on a collection of size six, there will exist an organized subsystem of size three. Showcased by examples:

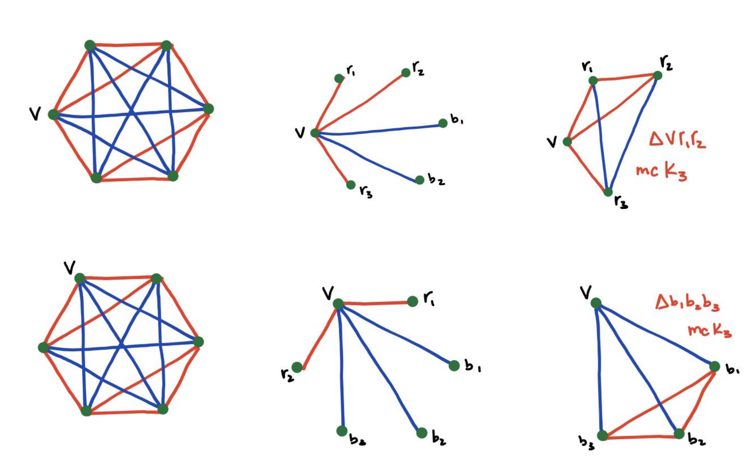


Party Problem Proof:
Our first goal is to show that among every group of 6 people there are 3 mutual friends or 3 mutual strangers. Let's consider person A - who is either friends or strangers with the remaining five people. Placing five people into two buckets, by PHP we are guaranteed at least three people in one of the buckets.

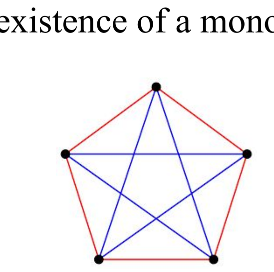


Thus, we know that person A is either friends with at least three people or strangers with at least three people.
Case 1: Person A is friends with three people. Among those three people, if any two people are friends, then we have found three mutual friends. If none are friends, then we have found three mutual strangers.
Case 2: Person A is strangers with three people. Among those three people, if any two people are strangers, then we have found three mutual strangers. If none are strangers, then we have found three mutual friends.
In both cases, we have either found three mutual strangers or three mutual friends, and we are done.

Graph Theory Proof : Given an arbitrary 2-coloring of the edges of K_6, select vertex $v \in V(K_6)$. Since there are 5 edges (pigeons) and 2 colors (holes), by PHP v is connected to at least 3 other vertices by either red or blue edges. WLOG, say the dominate color is red. Among the three vertices connected to v by red edges, either there exists a red edge and we've found a monochromatic K_3 in red, or there do not exist any red edges and we've found a monochromatic K_3 in blue.

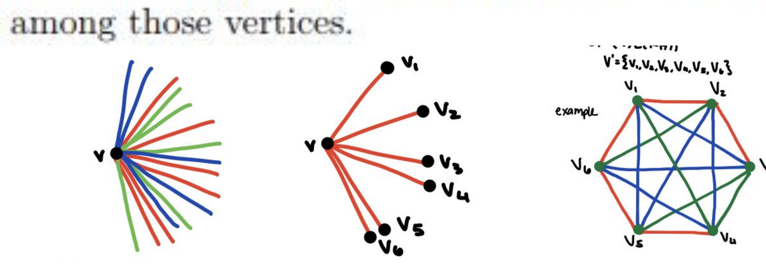


The two proofs above only give an upper bound, showing $R(3) \leq 6$. To prove equality, must show $R(3) \neq 5$. The 2-coloring of K_5 as constructed contains no monochromatic K_3, thus showing that 6 must be the smallest number of vertices required to ensure the existence of a monochromatic K_3.



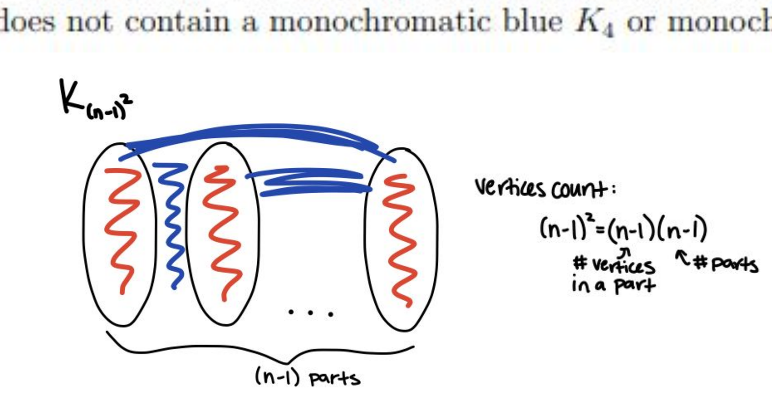
Deeper Problems

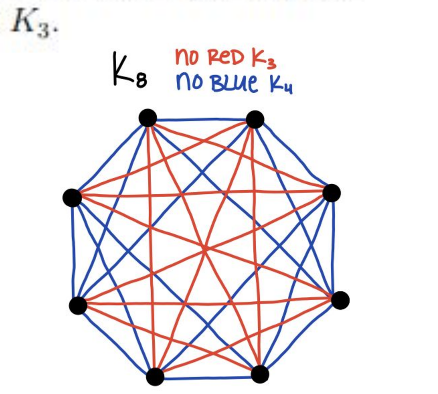
$\mathbf{R(3,3,3) \leq 17}$: Goal is to analyze an arbitrary 3-coloring of the edges of K_{17} and find a monochromatic K_3.
Pick vertex $v \in V(K_{17})$, which is connected to 16 other vertices with edges that are either red, blue, or green. Given 16 edges and 3 colors, $\lceil \frac{16}{3} \rceil = 6$, so by PHP there exists a color, say red, such that v is connected to six vertices by red edges. Looking at those six vertices, either there exists a red edge connecting two vertices, and thus we have found a red K_3, or no red edges exist. If no red edges exist among those six vertices, then the K_6 induced subgraph is 2-colored, and because $R(3) = 6$, we can conclude that either blue or green monochromatic K_3 exists among those vertices.



$\mathbf{R(n) > (n-1)^2}$: Idea is to show that there exists a 2-coloring of $K_{(n-1)^2}$ that does not contain a monochromatic K_n.
Proceeding by construction, partition $K_{(n-1)^2}$ into $n-1$ parts each containing $n-1$ vertices. Color the edges within each part red, and the edges connecting vertices in different parts blue. To achieve a red K_n requires one more vertex than contained in one part, and so one would need to venture beyond one part which picks up a blue edge. To achieve a blue K_n would entail selecting one vertex from distinct parts, and so cannot happen with only $n-1$ parts. Thus, this coloring of $K_{(n-1)^2}$ contains no red or blue K_n.

$\mathbf{R(3,4) \geq 9 \iff R(3,4) > 8}$: Idea is to show that there exists a 2-coloring of K_8 that does not contain a monochromatic blue K_4 or monochromatic red K_3.





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Attila Sali - Advanced Combinatorics Lectures, BSM CO2, Summer 2019
McGinley, Tamsen - Discrete Mathematics and Combinatorics Lectures, MATH 51&176, Fall 2018
Scott, Rick - Graph Theory Lectures, MATH 177, Spring 2019
Further Reading on Ramsey Numbers -- http://mathworld.wolfram.com/RamseyNumber.html

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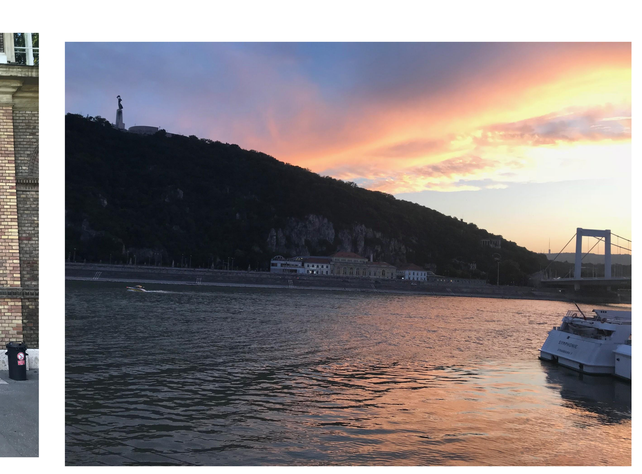

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Perforated Tableaux as Stembridge Crystals

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Perforated Tableaux

The following definitions explain **perforated tableaux** (1) as combinatorial objects of study. **Left justification** (2) is described as a process which moves content within the layout of the perforated tableau while continuing to satisfy column strictness. The **map *e_i*** (4) describes how content and blanks swap in the original tableau **T* in the resulting tableau *e_i(*T)*. Notation is included in this discussion to account for the **running count, maximum count**, and **left most maximum column**, which all play an important role in determining how *e_i* acts on **T* and consequently in verifying the Stembridge Crystal Axioms.

(1) Perforated Tableaux

Definition: *An rectangular tableau $T \in PTab_n$ will be called a perforated tableau if:*

1. *All entries in T are positive, or left blank (unfilled, denoted $\square$).*
2. *Positive integer content weakly increases in rows of T and strictly increases in columns of T.*
3. *For each $i \in [n]$, the i's appearing in T form a horizontal strip.*
4. *If $i, j \in [n]$ and $i < j$, every element in a horizontal strip of j's lies outside the northwest shadow of any entry in the horizontal strip of i's.*
5. *T has no column entirely composed of blanks (rows of all blanks are allowed).*

(2) Left-Justification

Definition: *Given some ptableau, T, we shall let $^*T$ denote the unique ptableau row equivalent to T in which row content is moved as far to the left as possible, the "left-justified form" of T. The process of left justification of a ptableau begins by shifting content to the left in the bottom row first, then moving sequentially to higher rows.*

Left justification of the ptableau T in the previous example yields the following. First the 1's in row 1, the horizontal strip of 2's remain fixed, then the 3 in row 3 slides left and the 4's in row 4 slide as far left as possible. Each step avoids violating column strictness.

$$T = \begin{array}{|c|c|c|c|c|c|}\hline & & & & 1 & 1 \\ \hline & 1 & 1 & & 2 & 2 \\ \hline 1 & 2 & 2 & 2 & & 3 \\ \hline 3 & 3 & & & 4 & 4 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline & & & 1 & 1 & \\ \hline & 1 & 1 & & 2 & 2 \\ \hline 1 & 2 & 2 & 2 & & 3 \\ \hline 3 & 3 & & & 4 & 4 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|}\hline & & & 1 & 1 & \\ \hline & 1 & 1 & & 2 & 2 \\ \hline 1 & 2 & 2 & 2 & 3 & \\ \hline 3 & 3 & & & 4 & 4 \\ \hline \end{array} \Rightarrow {}^*T = \begin{array}{|c|c|c|c|c|c|}\hline & & & 1 & 1 & \\ \hline & 1 & 1 & & 2 & 2 \\ \hline 1 & 2 & 2 & 2 & 3 & \\ \hline 3 & 3 & 4 & 4 & & \\ \hline \end{array}$$

(3) Covering Blanks and Covered Blanks

Definition: *In a ptableau T, if a blank is in row i with content directly below it, we say the blank is a* covering blank *in row i (the blank is covering content). If, however, a blank in row $i+1$ lies directly under content in row i, we shall call the blank a* covered blank *of row $i+1$ (the content covers the blank). With this, let*

$$\frac{\chi}{i+1}(j) = \text{The number of covering blanks of row } i \text{ in columns 1 through } j$$

and similarly

$$\frac{i}{\chi}(k) = \text{The number of covered blanks of row } i{+}1 \text{ in columns 1 through } k$$

(4) The map, *e_i*

Definition: *Given some i, for $1 \le i < n$, we define a map $e_i : PTab_n \to PTab_n \cup \{NULL\}$ as follows: Suppose T as col_T-many columns.*

Working from the left-justified form $^*T$ and reading the columns of $^*T$ from left to right, we intend to count the number of covering blanks up to column j against the number of covered blanks up to column $j-1$. Set $C(\frac{\chi}{i+1} - \frac{i}{\chi})(^*T, 1) = 0$ and then up to column j define

$$C\left(\frac{\chi}{i+1} - \frac{i}{\chi}\right)(^*T, j) = \left(\frac{\chi}{i+1}(j) - \frac{i}{\chi}(j-1)\right), \quad \text{for } j > 1$$

Then let

$$C\left(\frac{\chi}{i+1} - \frac{i}{\chi}\right)(^*T) = \max_{1 \le j \le col_T} C\left(\frac{\chi}{i+1} - \frac{i}{\chi}\right)(^*T, j)$$

and finally, set

$$m_i = \min\left\{k : C\left(\frac{\chi}{i+1} - \frac{i}{\chi}\right)(^*T) = C\left(\frac{\chi}{i+1} - \frac{i}{\chi}\right)(^*T, k)\right\}$$

If in column m_i there is a blank in row i and content in row $i+1$, then $e_i(T)$ is the perforated tableau obtained from $^*T$ by swapping the blank in row i of column m_i with the content in row $i+1$, column m_i. In all other cases, we let $e_i(T) = NULL$.

It is important to note that the effect of e_i on a ptableau is to move content from row $i+1$ up into row i by swapping the tiles creating a covering blank in the ptableau, when possible. We refer to $C(\frac{\chi}{i+1} - \frac{i}{\chi})(^*T, j)$, or more simply denoted $\chi_i(^*T, j)$, as the "running count" in row i of $^*T$ up to column j, as it keeps a tally of covering blanks against covered blanks at each previous column. The "maximum count" in row i is $C(\frac{\chi}{i+1} - \frac{i}{\chi})(^*T)$, also denoted ϵ_i, and occurs first in m_i, the "left-most max column."

An Example: Perforated Tableaux and *e_i* Operator

As an example, examine the perforated tableau **T* and calculate $e_2(^*T)$.

$$^*T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & & & & & 1 & 1 & 2 & 2 \\ \hline & & & & 1 & 2 & 2 & & 3 \\ \hline & & 1 & 1 & & 3 & 3 & 3 & \\ \hline 1 & 1 & 2 & 2 & 3 & 4 & 4 & 4 & \\ \hline \end{array}$$

$$^*T[2,3] = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & & & & 1 & 2 & 2 & & 3 \\ \hline & & 1 & 1 & & 3 & 3 & 3 & \\ \hline \end{array}$$

$$\chi_2(^*T, j) \quad \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ \hline \end{array}$$

After analyzing the running count, we can determine the max count in row 2, $\epsilon_2 = 2$
The left most max column is then found to be the fourth column, so $m_2 = 4$.
Therefore, when e_2 is applied to **T*, content in row 3 column 4 will swap up into row two.

$$\begin{array}{c} \\ ^*T = \end{array} \begin{array}{|c|c|c|c|c|c|c|c|c|} \multicolumn{3}{c}{} & \multicolumn{1}{c}{m_2} & \multicolumn{5}{c}{} \\ \multicolumn{3}{c}{} & \multicolumn{1}{c}{\downarrow} & \multicolumn{5}{c}{} \\ \hline & & & & & 1 & 1 & 2 & 2 \\ \hline & & & & 1 & 2 & 2 & & 3 \\ \hline & & 1 & 1 & & 3 & 3 & 3 & \\ \hline 1 & 1 & 2 & 2 & 3 & 4 & 4 & 4 & \\ \hline \end{array} \xrightarrow{e_2} e_2(^*T) = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & & & & & 1 & 1 & 2 & 2 \\ \hline & & & 1 & 1 & 2 & 2 & & 3 \\ \hline & & 1 & & & 3 & 3 & 3 & \\ \hline 1 & 1 & 2 & 2 & 3 & 4 & 4 & 4 & \\ \hline \end{array}$$

Stembridge Crystal Axioms

Axiom 2.1 *S0. If $e_i(T) = NULL$, then $\epsilon_i(T) = 0$*

Axiom 2.2 *S1. When $i, j \in I$ and $i \neq j$, if $T \in PTab$ and $e_i(T) \neq NULL$, then either $\epsilon_j(e_i(T)) = \epsilon_j(T)$ or $\epsilon_j(e_i(T)) = \epsilon_j(T) + 1$.*

Axiom 2.3 *S2. Assume that $i, j \in I$ and $i \neq j$. If $T \in PTab$ with $\epsilon_i(T) > 0$ and $\epsilon_j(e_i(T)) = \epsilon_j(T) > 0$, then $e_ie_j(T) = e_je_i(T)$ and $\phi_i(e_j(T)) = \phi_i(T)$.*

Axiom 2.4 *S3. Assume that $i, j \in I$ and $i \neq j$. If $T \in PTab$ with $\epsilon_j(e_i(T)) = \epsilon_j(T) + 1 > 1$ and $\epsilon_i(e_j(T)) = \epsilon_i(T) + 1 > 1$ then $e_je_i^2e_j(T) = e_ie_j^2e_i(T) \neq 0$, and $\phi_i(e_j(T)) = \phi_i(e_j^2e_i(T))$ and $\phi_j(e_i(T)) = \phi_j(e_i^2e_j(T))$*

Theorem: Perforated tableaux with the e_i operator form a combinatorial model for crystal structures that satisfies the Stembridge Crystal Axioms.

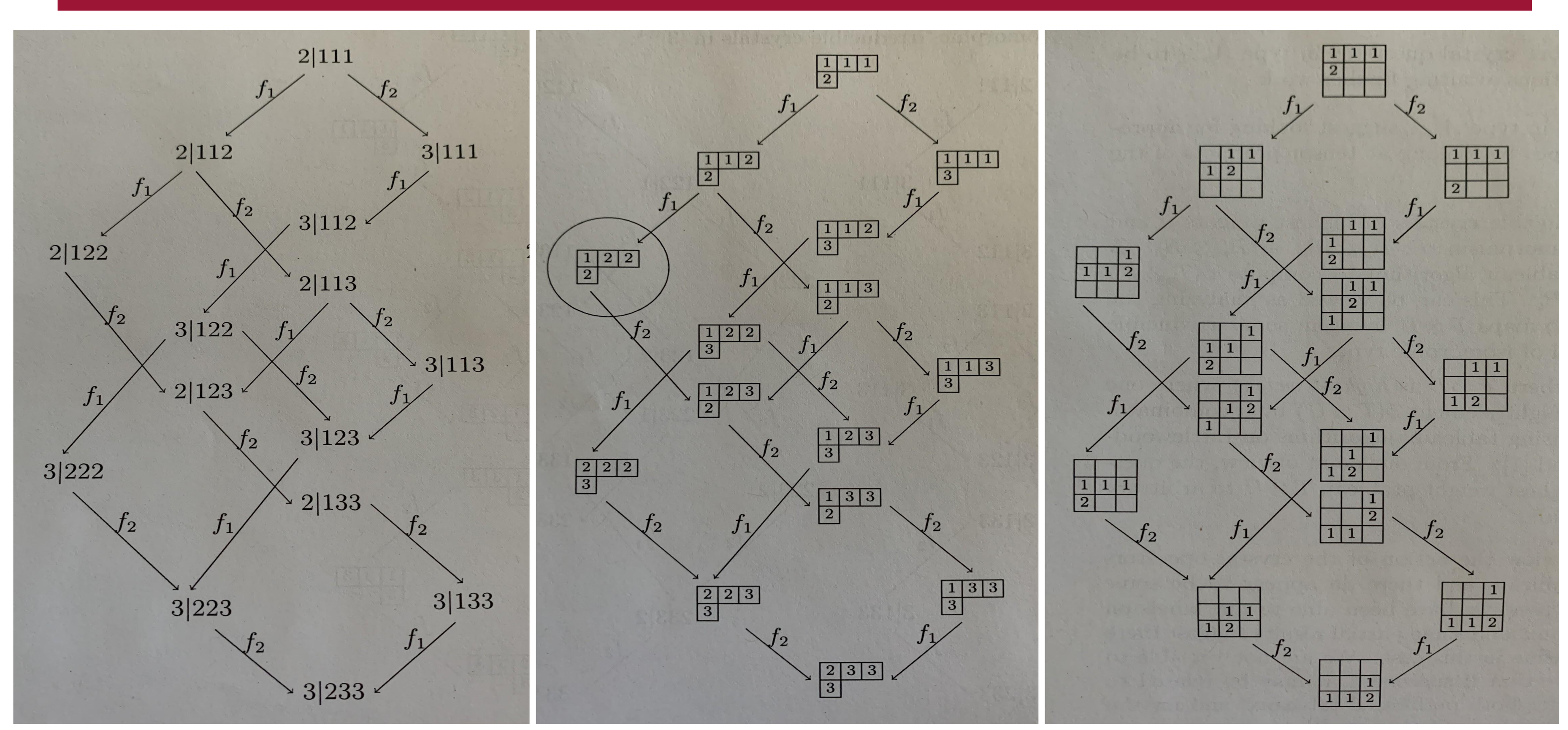


Figure of three models representing the same crystal: words (left), tableaux (middle), and perforated tableaux (right) (Appleby&McGinley)

One Of Many Cases

Proof of one case of Axiom 2

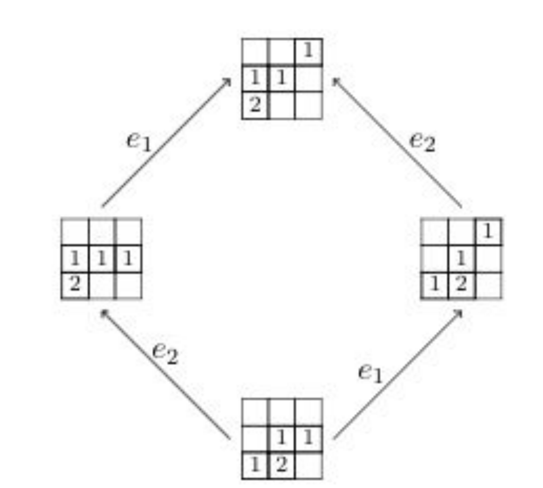
1. Let $j < i$ and $m_j < m_i$
   (a) Assume $j < i - 1$. Examine first the movement of content on the LHS, $e_ie_j(^*T)$. Since row i is below row j, no left justification occurs in row i in $^*e_j(^*T)$ and so the content layout remains as it was in $^*T$. Thus, m_i remains the left-most max column in $^*e_j(^*T)$.
   Next examine the RHS, $e_je_i(^*T)$. Since m_j is positioned to the left of m_i, the left justification of $e_i(^*T)$ does not impact column m_j, and so the content layout remains as it was in $^*T$. Thus, m_j remains the left-most max column in $^*e_i(^*T)$.
   (b) Assume $j = i - 1$. Examining the LHS, $e_i(e_j(T))$, we have the following layout:

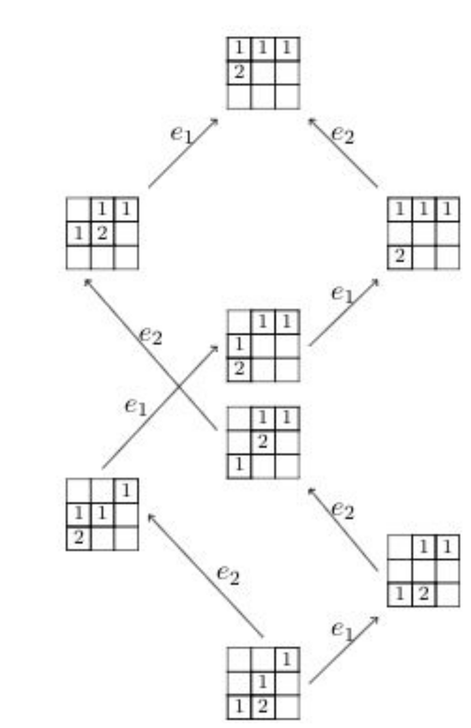
$$^*T : \begin{array}{|c|}\hline j \\ \hline i \\ \hline i{+}1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline m_j & \cdots & m_i \\ \hline b & \cdots & \\ \hline k & \cdots & b \\ \hline & \cdots & \ell \\ \hline \end{array} \xrightarrow{e_j} e_j(^*T) : \begin{array}{|c|}\hline j \\ \hline i \\ \hline i{+}1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline m_j & \cdots & m_i \\ \hline k & \cdots & \\ \hline b & \cdots & b \\ \hline & \cdots & \ell \\ \hline \end{array}$$

   By Axiom 1 Case 2, we know that the impact of e_j on $\epsilon_i(^*T)$ resulted in contributing one to the running count in row i. Since there is a blank in row i column m_i, left justification of row $j+1 = i$ in $e_j(^*T)$ terminates before or at column m_i. This implies that the addition to the running count in row i in $e_j(T)$ contributes a $+1$ to the max count at column m_i, and hence m_i remains the left-most max column in $^*e_j(^*T)$. Since content in row $i+1$ does not shift under left justification of $e_j(^*T)$, $e_i(e_j(^*T))$ moves the same content as $e_i(T)$.
   Examining now the RHS $e_j(e_i(^*T))$, we first have content in column m_i swapping. By hypothesis, $\epsilon_j(e_i(^*T)) = \epsilon_j(^*T)$, which implies that the addition of content into row $i = j+1$ does not increase the maximum count in row j. Therefore, m_j remains the left-most max column in row j. Content in column m_j is unaffected by left justification of $e_i(^*T)$ since it is positioned to the left of m_i, and thus content swapped in column m_j under $e_j(^*e_i(^*T))$ is the same as that swapped in $e_j(^*T)$.

Therefore, for $j < i$ and $m_j < m_i$, we have $e_i(e_j(T)) = e_j(e_i(T))$.

Crystal Substructures





Axiom 2 shown above
Axiom 3 shown to the right

In the square diagram, we can examine Axiom 2 in action. Taking the tableau at the bottom to be $^*T$, we calculate $\epsilon_2(^*T) = 1$ and $\epsilon_2(e_1(^*T)) = 1$. Then by Axiom 2, we know that $e_1(e_2(^*T)) = e_2(e_1(^*T))$. Similarly in the figure eight diagram we can observe Axiom 3. Again taking the tableau at the bottom as the original $^*T$, we have

$$\epsilon_1(^*T) = 1 \text{ and } \epsilon_1(^*T) = 1$$

$$\epsilon_1(e_2(^*T)) = 2 \text{ and } \epsilon_1(e_1(^*T)) = 2$$

Therefore, since $\epsilon_1(e_2(^*T)) = \epsilon_1(^*T) + 1 > 1$ and $\epsilon_2(e_1(^*T)) = \epsilon_2(^*T) + 1 > 1$, we conclude that $e_ie_j^2e_i(^*T) = e_je_i^2e_j(^*T)$.

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