DRP: Lattices and Polytopes

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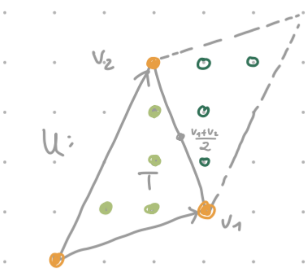
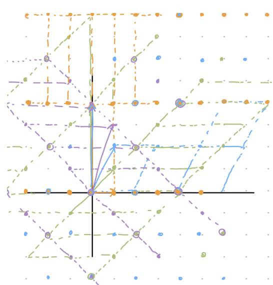
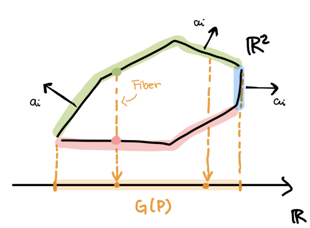
Seminar Description: This seminar will be an introduction to the theory of lattices and polytopes. You can think of a polytope as what happens when you take points in space and snap a (dimension appropriate) rubber band around them. So polytopes are a generalization of forms like a triangle, cube, prisma, pyramid, tetrahedron, etc. They are a geometric way to realize optimization problems from the real world and within mathematics, many other problems can be reformulated into the theory of polytopes. This, below, is what doing math with polytopes looks like!



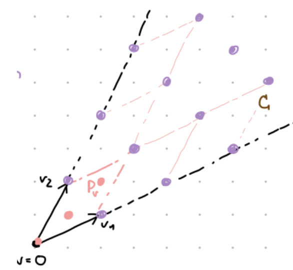
(a) Pick's Theorem



(b) lattice bases



(c) projecting polytopes



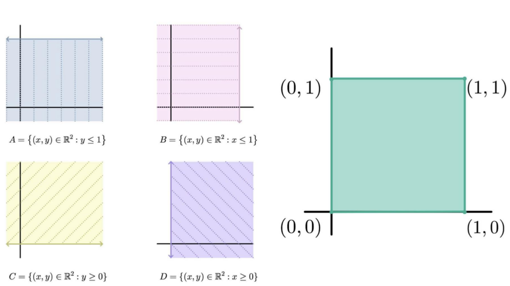
(d) counting lattice points

Prerequisites: Mathematical reasoning, vectors, $\mathbb{R}^n$, functions and polynomials (MAT 108, MAT 22A)

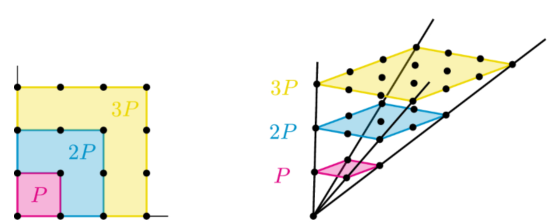
Texts: Computing the Continuous Discretely (Beck, Robins), and our seminar notes

Meeting Format and Expectations: We are planning to meet for ***90 minute*** sessions once a week and we will do our best to pick a date/ time that works for everyone. Each session will begin with a small lecture to help everyone get to know the concepts. After this, we will work on problems designed to expand and reinforce concepts in small groups. During this time, we will ask students to work collaboratively to fill in details of proofs, play with some fun examples, and solidify their understanding. We believe that mathematics is best learned by *doing mathematics* so we will provide many opportunities to participate. There will be optional take home problems (i.e., homework) for students to work on if they wish to dive deeper and learn more advanced concepts. These are entirely optional and we hope to keep the time commitment outside of seminar meetings minimal.

Learning Goals: basic understanding of lattices and polytopes (of any dimension), relation of lattice points and volume, application to linear programming, generating functions, Ehrhart's theorem for integral polytopes



(a) Multiple Ways to Describe Polytopes



(a) (b)

(b) Key Idea of Ehrhart Theory

Here is a rough outline of what we plan to cover each week.

Lesson Planning:

1. **Basic introduction:** review of terminology and notation, introduction
2. **Introduction to polytopes:** definitions, inner and outer description of polytopes, faces, vertices, examples
3. **Why are polytopes useful? Application to Linear Programming:** setting up a linear program and the relation to polytopes, feasible solutions, optimal solutions, examples
4. **Lattice crashcourse:** lattices, bases, fundamental domain (determinant), lattice polytopes, examples
5. **Pick's Theorem (with proof):** comparison between lattice points and volume, examples in $\mathbb{R}^2$, proof of Pick's Theorem, motivation that "counting lattice points is a polynomial"
6. **Preparation for Ehrhart Theory Part 1:** introduction to generating functions, generating functions and arithmetic, using generating functions for combinatorial arguments, quasi-polynomials
7. **Preparation for Ehrhart Theory Part 2:** introduction to ideas used in the proof for lattice polytopes: triangulations, cones, the lattice point generating function, inclusion-exclusion, the parallelepiped
8. **Ehrhart's Theorem:** statement for lattice polytopes, examples, sketch of proof
9. ***(Optional)*** **More Ehrhart's Theorem:** Ehrhart's theorem for rational polytopes applied to magic squares and numerical semigroups.