Weinstein skeleta

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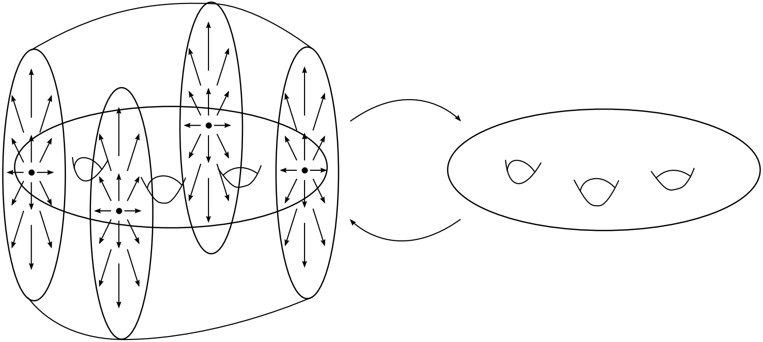
The goal

$$\{(W,\omega) \text{ symplectic geometry }\} \rightleftarrows \{\text{Skeleton (smooth and singular topology)}\}$$

$$(T^*M, \omega_{can}) \rightleftarrows M$$



Exact symplectic structure

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Weinstein manifolds

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Weinstein manifolds

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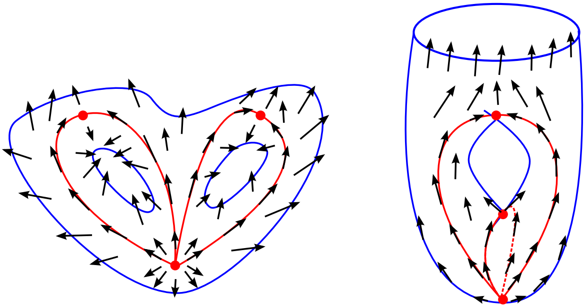
Convexity condition: V points outward along the boundary
(in the noncompact case, this should be true for a compact exhaustion)

Weinstein condition: V is *gradientlike*.

$$V = \nabla_g \phi$$

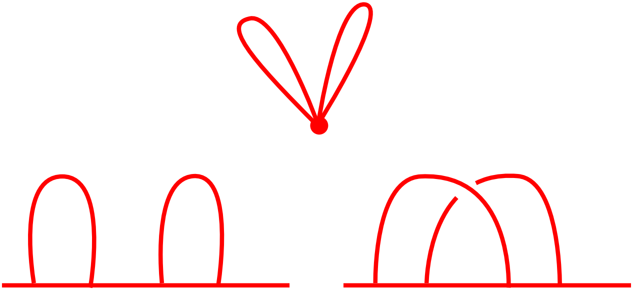
Skeleta and singularities

Skeleton: The points in W which do not escape to the boundary under the flow of V

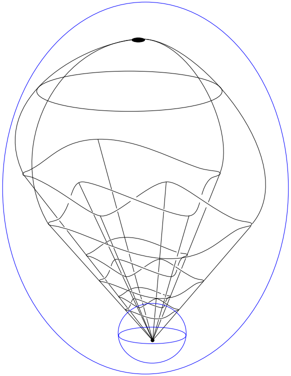


Weinstein case: The skeleton is *Lagrangian/isotropic*.

Skeleta built of bones



Skeleta and singularities



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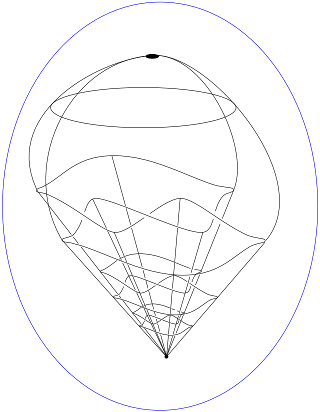
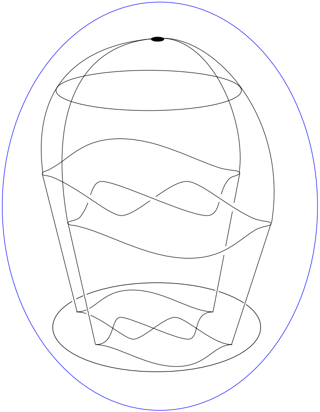




Skeleta built of bones

Theorem (S. 2017)

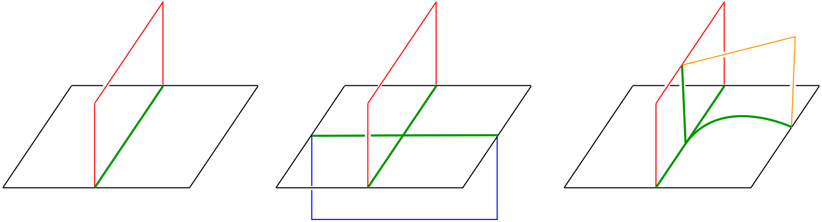
1. *Every Weinstein manifold* (W, ω, V) *has a deformation of* V *such that the skeleton is built of bones.*
2. *The bones (manifolds with boundary) and the joints between them (singular hypersurfaces of the bones) uniquely determine* (W, ω).

All the *symplectic geometry* of the $2n$ dimensional manifold W, can be recovered from the *singular topology* of the n-dimensional skeleton.

Controlling singularities

Next goal: Have combinatorial models for all the singularities we need in the skeleton

Nicest class of singularities in skeleta built of bones: immersed joints



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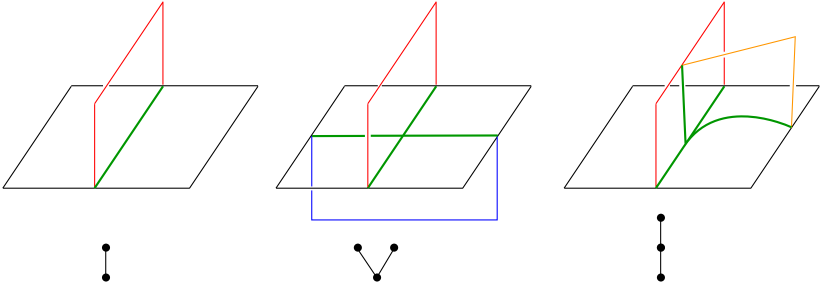

These coincide with Nadler's *arboreal singularities*

Arboreal skeleta

What if the joints are not immersed?

Deform V more to break the skeleton into more bones to eliminate the non-immersed singularities.

I have done this for the first type of non-immersion points, which leads to:

Theorem (S. 2017)

All Weinstein 4-manifolds, after deforming the Weinstein structure V, have a skeleton built of bones with generically immersed joints (thus arboreal singularities).

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More generally

The next stage: Inductive step to utilize the elimination of the first type of non-immersed joints to eliminate all types of non-immersion points.

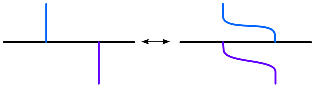
Work in progress (Eliashberg-Nadler-S.)

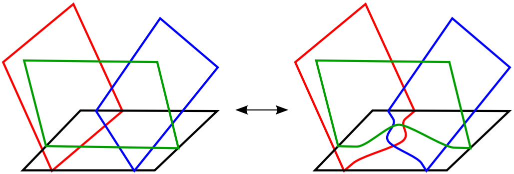
Every Weinstein manifold has a deformation of V such that the skeleton is built of bones with immersed joints (arboreal singularities).

Going forward

Arboreal moves: we know there is a skeletal calculus of finitely many moves on an arboreal skeleton which gets between any two different skeleta of the same (W, ω).

Next goal: list the moves explicitly, and extract invariants.





Thank you!!