1 Lecture Review

So there are two concepts that you’ve covered since our last discussion: double integrals in polar form, and triple integrals in rectangular coordinates. The double integral in polar form is given by first changing from the $(x, y)$ coordinate system to the $(r, \theta)$ coordinate system. This is

$$(x, y) = (r \cos(\theta), r \sin(\theta))$$

But to do so, we need a way to change the differential $dx
dy$ into $dr\,d\theta$. This transformation is given in terms of what’s call the Jacobian, or total derivative, of the function $\vec{F}(r, \theta) = (x, y) = (r \cos \theta, r \sin \theta)$:

$$dxdy = |J|dr\,d\theta$$

where the vertical bars mean we’re taking the determinant of the matrix$^1$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial x}{\partial r} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r} = r \cos^2 \theta + r \sin^2 \theta = r$$

So a 2D integral of a function $f(x, y)$ over a region $D$ becomes

$$\int \int_D f(x, y) \, dx\,dy = \int \int_D f(r \cos \theta, r \sin \theta) r \, dr\,d\theta = \int \int_D f(r \cos \theta, r \sin \theta) r \, d\theta\,dr$$

It’s worth noting that we can still change the order of integration, in case it makes the problem easier.

The second section you covered was rectangular triple integration. The only real mathematical complication that we’ve added is that the innermost bounds of integration now have 2 parameters they can depend on:

$$\int \int \int_{x_0, y_0(x), z_0(x,y)} F(x, y, z) \, dx\,dy\,dz$$

that being said, figuring out the bounds can be considerably trickier in 3D.

2 Example problems

I suggest focusing on one of these; if you finish it early, then try another. They’re ordered roughly by difficulty.

Example 2.1. 15.5.17 Compute the value of the integral

$$\int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{0} \frac{2}{1 + \sqrt{x^2 + y^2}} \, dy\,dx$$

$^1$If you haven’t seen determinants before, don’t worry, you won’t be tested on this general form. I wanted to include it for completeness sake. If you’d like to learn more, I have links on my teaching page which have tutorials on linear algebra. Or you could read Wikipedia.
Example 2.2. 15.5.15(modified) Compute the value of the integral

\[ \int_{0}^{1} \int_{0}^{(2-x)(2-x-y)} \int_{0}^{z} dz \, dy \, dx \]

Example 2.3. Let \( D \) be some half-circle with radius \( \frac{\pi}{2} \). Find the integral

\[ \int_{D} \cos \left( \frac{x^2 + y^2}{\pi} \right) \, dA \]

Example 2.4. Let \( D \) be a cube with edge length \( a \), with the origin at the center of the cube (that is \( D = \{ \vec{x} \mid -a/2 \leq x \leq a/2, -a/2 \leq y \leq a/2, -a/2 \leq z \leq a/2 \} \))

\[ I_x = \int_{D} \int \int \delta(y^2 + z^2) \, dV \]

This can be interpreted as the moment of inertia of a cube with density \( \delta \) about the \( x \) axis (or any principal axis (that is, the \( x \), \( y \), or \( z \) directions)).

Example 2.5. Evaluate the integral

\[ \int_{0}^{2} \int_{0}^{1} \frac{4 \cos(x^2)}{2\sqrt{z}} \, dx \, dy \]

(hint you may want to think about changing the order of integration)

Example 2.6. Prove that the Gaussian integrates to \( \sqrt{\pi} \). That is, prove that

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \]

(hint: think about integrating the function \( e^{-x^2-y^2} \) over the whole plane; how could you use polar integration to solve this? Also, how does it relate to the integral of \( e^{-x^2} \)?)

Example 2.7. Find the volume of \( D \), the solid bounded below by the \( xy \)-plane, and above by the surface \( z - 1 + x^2 + y^2 = 0 \). You may want to express \( x \) and \( y \) in polar coordinates.