GKM spaces, and the signed positivity of the nabla operator

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October 14, 2021

Abstract

We show that the Frobenius character of the equivariant Borel-Moore homology $H_*^T(Z_k)$ of a certain positive GL_n-version of the unramified affine Springer fiber studied by Goreski, Kottwitz and MacPherson [GKM04] is computed by the matrix coefficients of the ∇^k-operator, which acts diagonally in the modified Macdonald basis. We do this by relating the combinatorial formula of [CM20] for the ∇^k-operator to the GKM paving of Z_k, and we give an algebraic presentation of the above homology as an explicit submodule of the Kostant-Kumar nil Hecke algebra [KK86]. We then study a certain open locus $U_k \subset Z_k$, and reduce a long-standing conjecture of [BGHT99], which predicts the sign of the coefficients of the Schur expansion of ∇^k, to a vanishing conjecture about the homology groups of U_k. The latter conjecture is in turn reduced to a vanishing conjecture for certain open loci of the regular semisimple Hessenberg varieties which are indexed by partial Dyck paths.

1 Introduction

Let ∇ be the nabla operator of [BGHT99], which is diagonal in the basis of modified Macdonald polynomials. Let X, Y be alphabets for two different sets of symmetric functions, and let $\nabla = \nabla_X$ act in the X-variables. We adopt the usual plethystic notation of [Hai01b], in which $f[A]$ is the result of substituting $p_k = A|_{v=v^k}$ where v ranges over all variables that appear in A. In [CM20], we proved

Theorem 1.1. *For any $k \geqslant 1$, we have*

$$\nabla^k e_n \left[\frac{XY}{(1-q)(1-t)}\right] = \sum_{[\mathbf{m},\mathbf{a},\mathbf{b}]} \frac{t^{|\mathbf{m}|} q^{\mathrm{dinv}_k(\mathbf{m},\mathbf{a},\mathbf{b})}}{(1-q)^n \mathrm{aut}_q(\mathbf{m},\mathbf{a},\mathbf{b})} X_{\mathbf{a}} Y_{\mathbf{b}}. \tag{1}$$

Here $\mathbf{m} \in \mathbb{Z}_{\geqslant 0}^n$, $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geqslant 1}^n$ are labels for the variables $X_{\mathbf{a}}, Y_{\mathbf{b}}$, and $[\mathbf{m}, \mathbf{a}, \mathbf{b}]$ denotes an orbit under the S_n action by simultaneously reordering the elements. The absolute value $|\mathbf{m}|$ is just the sum of the elements, $\mathrm{dinv}_k(\mathbf{m}, \mathbf{a}, \mathbf{b})$ is a certain combinatorial statistic extending the one by the same name that appears in the Shuffle Theorem [HHL+05, HMZ12, CM18], and $\mathrm{aut}_q(\mathbf{m}, \mathbf{a}, \mathbf{b})$ is an explicit product of q-factorials.

The dinv statistic in equation (1) is essentially the dimension of certain Schubert cell in the affine Springer fiber Y_k of [GKM97, GKM03, GKM04], described in Section 4.4. The Frobenius character and the automorphism factors were discovered by studying an explicit module M_k over the polynomial ring $\mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$, which was conjectured to be the image of $P \otimes \mathcal{O}(k)$ under the Haiman-Bridgeland-King-Reid isomorphism [Bri07, Hai01b, Hai01a]. As we have recently learned, this has been shown using some new results in [AL21] using Cherednik algebras. The results of [CM20] did not attempt to connect with Haiman's theory, and instead verified equation (1) by establishing that the expected sum satisfies the defining properties of the ∇^k operator, namely triangularity, symmetry, and leading term. An alternative proof was given by interpreting the sum as an enumeration over parabolic vector bundles over $\mathbb{P}^1$, which had been related to Macdonald polynomials in [Mel20].

1.1 The affine Springer fiber

We describe now the geometric picture. Let $G = GL_n$, with standard maximal torus $T \subset G$, which has Weyl group S_n. The GL_n affine flag variety Y is the quotient $G(\mathcal{K})/I$ where $\mathcal{K} = \mathbb{C}((z))$ and I is the stabilizer of the standard infinite flag of lattices in $\mathcal{K}^n$. It has a structure of an ind-variety, and its connected components are in bijection with integers, each one being isomorphic to its SL_n version. We have an action of the $(n+1)$-dimensional torus $\tilde{T} \supset T$ which includes the loop rotation parameter, and the fixed points are in bijection with the affine Weyl group W, which is the set of (extended) affine permutations. The Schubert decomposition $G(\mathcal{K}) = \bigsqcup_{w \in W} IwI$ corresponds to the paving of Y by I-orbits Ω_w, which are isomorphic to affine spaces. The set of affine permutations W is endowed with Bruhat order so that any lower set of W corresponds to a closed subvariety of Y. The set of positive affine permutations $W_+ \subset W$, which take positive integers to positive integers, is an example of a lower set, and thus corresponds to a closed subvariety which we denote by $Z \subset Y$. Every connected component of Z consists of a finite number of cells, and therefore is an ordinary variety.

Let $Y_k = Y_{\gamma_k} \subset Y$ denote the unramified affine Springer fiber studied in [LS91, GKM04] in type A, associated to the topologically nilpotent element

$$\gamma_k = \mathrm{diag}(a_1 z^k, ..., a_n z^k),$$

where $a_i \in \mathbb{C}^*$ are distinct, which is simply the set of points of Y fixed by $1 + \gamma_k$. It is shown in [GKM03] that Y_k is paved by affine spaces given by the intersections $Y_k \cap \Omega_w$. The torus T as well $\tilde{T}$, act on Y and preserve Y_k, the cells Ω_w, and Y^+, and so they also acts on the intersection $Z_k = Y_k \cap Z$, which we call *the positive part*. The T-equivariant Borel-Moore homology is denoted by H_*^T and is a module over $S = H_T^*(pt) = \mathbb{C}[x_1, ..., x_n]$, where the x_i are the generators of Lie(T). We use the convention the x_i are positively graded, forcing homological degree to be graded in the negative direction, so that the corresponding Frobenius character is a formal Laurent series in q. We have inclusions

$$H_*^T(Y_k) \subset H_*^T(Y) \subset F \otimes_S H_*^T(Y^T) \cong F \cdot W, \tag{2}$$

where $F = \mathbb{C}(x_1, ..., x_n)$ is the field of fractions of S, $F \cdot W = F \otimes \mathbb{C}[W]$. These spaces are GKM with respect to a larger torus $\tilde{T} \supset T$ which includes the loop rotation parameter, which means that $H_*^{\tilde{T}}(Y_k)$ may be described by explicit relations in $\tilde{F} \cdot W_+$.

The non-affine Weyl group S_n acts on $F \cdot W$ on the left and on the right. The left action is called the *dot action* and it intertwines the action of S, whereas the right action is called the *star action* and it commutes with S. Both of these actions descend to $H_*^T(Y_k)$. The dot action on $H_*^T(Y_k)$ becomes identified with the space level action of S_n on Y by the left multiplication. Multiplication by $w \in W$ sends Y_{γ_k} to $Y_{w(\gamma_k)}$, so we do not have a space level action per se, but the homology of Y_k for different choices of γ_k are all naturally identified via the parallel transport, or alternatively via the embedding $H_*^T(Y_k) \subset H_*^T(Y)$.

The star action on the other hand is the Springer action, and can be understood as follows. For any parabolic $P_\alpha \supset B$ we have versions of the spaces Y_k, Y where we use I_α, which is the subgroup of $G(\mathcal{K})$ preserving the corresponding standard partial flag of lattices of $\mathcal{K}^n$ instead of I, and is an example of a *parahoric* subgroup. Denote these spaces by $Y_k^\alpha \subset G(K)/I_\alpha$. The natural projections

$$Y_k \to Y_k^\alpha, \quad Y \to Y^\alpha$$

are proper and T-equivariant, and therefore induce pushforward maps on H_*^T. The Springer action is the unique action which identifies these pushforward maps with the projection maps to S_α-invariants, where $S_\alpha = S_{\alpha_1} \times \cdots \times S_{\alpha_l} = S_n \cap P_\alpha$ is the corresponding Young subgroup.

Both actions descend to $H_*^T(Z_k) = H_*^T(Y_k) \cap F \cdot W_+$.

We may now consider the bigraded Frobenius character. We encode the Frobenius character of the star and dot action by symmetric functions in the alphabet X and Y

respectively. We have that $H_*^T(Z_k)$ is bigraded so that the first grading is half of the cohomological grading, i.e. negative of the homological grading, which exists only in even degree, and is encoded by powers of the variable q. The second is the space-level grading by the index of the connected component of Z_k, which is nonempty only for nonnegative integers, encoded by powers of t.

Theorem A. *We have*

(a) *The equivariant homology of the Springer fiber is identified with an explicit submodule of the Kostant-Kumar nil-Hecke algebra*

$$H_*^T(Y_k) = \widehat{\mathbb{A}}_{af} \cap \Delta(\mathbf{x})^{-k} S \cdot W,$$

where $\Delta(\mathbf{x}) = \prod_{i<j}(x_i - x_j)$, *and the intersection is taken in* $F \cdot W$.

(b) *The Frobenius character of the positive part is given by*

$$\mathcal{F}_{Y,X} H_*^T(Z_k) = q^{-k\binom{n}{2}} \omega_X \nabla^k e_n \left[\frac{XY}{(1-q)(1-t)}\right]. \tag{3}$$

In part (b), which we explain first, ω_X denotes the involution $s_\lambda \to s_{\lambda'}$ applied with respect to the X variables. This result is proved using Theorem A from [CM20], by establishing that the right side of (1) computes the Frobenius character. Without the dot action, the right hand side is just counting cells in the affine paving of the parabolic version of Y_k using the Springer action, with the number $k\binom{n}{2} - \mathrm{dinv}_k$ being the dimension of a cell. To compute the dot action, we use the alternative presentation of (1) as a sum of certain LLT polynomials $\xi_\pi[Y;q]$, which are the Frobenius characters of the regular semisimple Hessenberg varieties by a conjecture of Shareshian and Wachs, proved in [BC15, SW16]. We then verify that our decomposition into LLT polynomials precisely matches the paving of Y_k by affine bundles over Hessenberg varieties from [GKM03], and restricting this paving to Z_k we obtain the result. The automorphism factors appear because the fibers of the flag versions of these Hessenberg varieties over certain parabolic versions are products of usual flag varieties.

In part (a), we identify $H_*^T(Y_k)$ with an explicit bigraded $S_n \times S_n$-module which we now define. The group of affine permutations W is generated by the group of finite permutations S_n and the rotation element $\rho_i = i + 1$. We have the action of Kostant and Kumar's nil Hecke algebra $\mathbb{A}_{af}$ on the F-vector space $F \cdot W$ with basis vectors labeled by p_w for $w \in W$, see [KK86, LLM+14]. The submodule

$$\widehat{\mathbb{A}}_{af} = \sum_{d \in \mathbb{Z}} \mathbb{A}_{af} \rho^d \subset F \cdot W$$

is free over $\mathbb{A}_{af}$, and is an algebraic presentation of the equivariant Borel-Moore homology of the affine flag variety Y for GL_n with the action of the small torus $T \subset GL_n$. The intersection in the first part may now be understood as the S-submodule consisting of elements of $\widehat{\mathbb{A}}_{af}$ whose denominator divides $\Delta(\mathbf{x})^k$.

1.2 An open subvariety and nabla positivity

Finally, we present a potential application which can be seen as a categorification of the famous nabla positivity conjecture of [BGHT99], which states that the coefficients $c_{\lambda,\mu}(q,t)$ of the Schur expansion

$$(-1)^{\iota(\lambda')}\nabla^k s_\lambda = \sum_\mu c_{\lambda,\mu}(q,t) s_\mu$$

are nonngegative, and $\iota(\lambda)$ is the number of boxes of λ below the main diagonal (see Conjecture 3.1).

In our approach, we consider the action of $\mathbb{C}[y_1, ..., y_n] = \mathbb{C}[\mathbb{Z}^n_{\geqslant 0}]$ on $H^T_*(Z_k)$ by the left (dot) multiplication by

$$y_i \in \mathbb{Z}^n_{\geqslant 0} \subset W_+ \subset W,$$

which sends $i \mapsto i + n$, and fixes the other elements of $\{1, ..., n\}$. These are induced from the restriction of the space-level action of the translation group $\mathbb{Z}_{\geqslant 0} \subset \mathbb{Z}^n \subset W$ on Z_k. We thus obtain that $H^T_*(Z_k)$ is a bigraded $S_n \times S_n$ equivariant module over $R = \mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$, in which the left action of S_n intertwines the diagonal action on R.

We then prove that the action of $\mathbb{C}[y_1, ..., y_n]$ is free, which is analogous to Haiman's freeness of polygraph rings over one set of variables on the "coherent side" of the BKR ismorphism [Hai01a, Hai01b], and we have learned would follow from the results of [AL21], once one has the identification of M_k from Theorem A. We give two different proofs of the freeness. One proof is geometric: we study a certain family of subvarieties $Z^i_k \subset Z_k$, which we prove are S_n-translates of unions of intersected Schubert cells, and satisfy $H^T_*(Z^i_k) = y_i H^T_*(Z_k)$ via the inclusion map. We then show that their homologies satisfy a "distributive lattice" type property in Proposition 6.7 and conclude that the y_i define a regular sequence on $H^T_*(Z_k)$.

Another proof uses explicit symmetry between x and y variables, which may be interesting on its own, see Theorem 6.2. This symmetry is a consequence of a GL_2 action constructed with the the help of explicit differential operators using the presentation of $H^T_*(Z_k)$ as the intersection $\widehat{\mathbb{A}}_{af} \cap \Delta(\mathbf{x})^{-k} S \cdot W_+$. It turns out our operators preserve both $\widehat{\mathbb{A}}_{af}$ and $\Delta(\mathbf{x})^{-k} S \cdot W_+$.

We then consider the quotient module $H_*^T(Z_k)/(y_1,...,y_n)H_*^T(Z_k)$, for which we also give a geometric interpretation: let $U_k = Z_k - Z$ for $Z = Z_k^1 \cup \cdots \cup Z_k^n$ be the complement. Then we have the long exact sequence

$$\cdots \to H_*^T(Z) \to H_*^T(Z_k) \to H_*^T(U_k) \to H_{*-1}^T(Z) \to \cdots$$

Using the distributive lattice property, we show that this splits into short exact sequences, so that $H_*^T(U_k)$ is supported in even degrees, and we have that (Corollary 6.10)

$$H_*^T(U_k) \cong H_*^T(Z_k)/(y_1,...,y_n)H_*^T(Z_k).$$

Note that this does not imply that U_k is equivariantly formal, only that *equivariant* Borel-Moore homology is supported in even degree. Indeed, equivariant formality would imply that $H_*^T(U_k)$ is free over $\mathbb{C}[x_1,...,x_n]$ which is untrue, and in fact U_k has odd non-equivariant homology.

Then the ith Tor group $\mathrm{Tor}_i^R(H_*^T(Z_k),\mathbb{C}) \cong \mathrm{Tor}_i^S(H_*^T(U_k),\mathbb{C})$ inherits the left and right action of S_n as well as the bigrading. We propose the following categorification of nabla positivity:

Conjecture A. The multiplicity of the irreducible representation χ_λ of the left action of S_n on $\mathrm{Tor}_i^R(H_*^T(Z_k),\mathbb{C}) \cong \mathrm{Tor}_i^S(H_*^T(U_k),\mathbb{C})$ is nonzero for at most one value of i, namely $i = \iota(\lambda)$.

The equivariant homology $H_*^T(U_k)$ is pure, and therefore by [FW05] the Tor groups above can be interpreted as steps in the weight filtration in the corresponding non-equivariant homology $H_*(U_k)$. For instance, Tor_0 corresponds to the pure part of the homology, Tor_1 to the part one degree off from pure, and so on, see Corollary 6.15.

We further refine Conjecture A by intersecting U_k with the the paving by affine bundles over Hessenberg varieties of [GKM03]. Interestingly, while the Hessenberg varieties from this paving are labeled by Dyck paths, their intersections with U_k have slightly more structure, and in fact are determined by *partial* Dyck paths, in which the number of missing steps l corresponds to the number of nontrivial intersections with the Z_k^i. The homology groups of these varieties $H_*^T(U_{\pi,l}) \cong H_T^*(U_{\pi,l})$, where the second isomorphism follows by smoothness of $U_{\pi,l}$, are further conjectured to satisfy the vanishing property in the following refinement of Conjecture A:

Conjecture A′. Conjecture A holds with $U_{\pi,l}$ in place of U_k.

Again, $H_*^T(U_{\pi,l})$ is pure, and therefore the conjecture can be interpreted in terms of the weight filtration on $H_*(U_{\pi,l}) \cong H^*(U_{\pi,l})$, see Corollary 6.20.

We then prove our second result:

Theorem B. *Conjecture A implies the nabla positivity conjecture of [BGHT99]. Conjecture A′ implies Conjecture A.*

Both conjectures A and A′ are supported by algebraic calculations using Gröbner bases in SAGE and MAPLE. Additionally, in Proposition 6.25 we prove an explicit formula for the Frobenius character of $H_T^*(U_{\pi,l})$ as a quasisymmetric function, which after the plethystic substitution $Y \to (1-q)Y$ may be readily observed on a computer to satisfy the signed positivity property for large values of n, for instance up to at least 10. In an upcoming paper, we deduce this fact from an explicit combinatorial formula for the Schur expansion of $\mathcal{F}_Y H^*(U_{\pi,l}) = \mathcal{F}_{(1-q)Y} H_T^*(U_{\pi,l})$, which also implies the Loehr-Warrington formula [LW08] and establishes the signed positivity of $\nabla^k s_\lambda$ in the monomial basis m_μ.

2 Notations

For convenience of the reader, we summarize below some notation used throughout the paper.

G	$= GL_n$
T	$= (\mathbb{C}^*)^n \subset G$, the small torus
$\widetilde{T}$	$= T \times \mathbb{C}^* = (\mathbb{C}^*)^{n+1}$ the big torus
S	$= \mathbb{C}[x_1, \ldots, x_n] = H_T^*(\text{point})$
$\tilde{S}$	$= \mathbb{C}[x_1, \ldots, x_n, \epsilon] = H_{\tilde{T}}^*(\text{point})$
$F, \tilde{F}$	the fraction field of S, $\tilde{S}$ respectively
S_n	the permutation group, i.e. the Weyl group of G
W	the group of affine permutations, i.e. the extended affine Weyl group
W_+	the monoid of positive affine permutations
Y, Y_k	the affine flag variety, the affine Springer fiber respectively
Z, Z_k	the positive part of Y, Y_k
H_T^*	the equivariant cohomology
H_*^T	the equivariant Borel-Moore homology.

Acknowledgments. E. Carlsson was supported by NSF DMS-1802371 during part of this project. A. Mellit was supported by the projects Y963-N35 and P31705 of the Austrian Science Fund.

3 Combinatorial background and preliminaries

We recall some basic combinatorial background and notation, and also the main theorem of [CM20], which is Theorem 1.1 from the introduction.

3.1 Symmetric functions

Let Λ denote the ring of symmetric functions, and write Λ_R if we would like to specify a ground ring R, which by default is equal to $\mathbb{Q}$. We have the usual bases $s_\lambda, m_\lambda, h_\lambda, e_\lambda, p_\lambda$ labeled by partitions $\lambda \in \operatorname{Par}(n)$, see [Mac95]. For $f \in \Lambda$, let $f[A]$ be the plethystic substitution homomorphism, which evaluates f at $p_k = A^{(k)}$, which is the result of substituting $v = v^k$ for every variable appearing in A, as in [Hai01b].

If V is a representation of S_n, we have its Frobenius character

$$\mathcal{F}V = \sum_{\lambda \in \operatorname{Par}(n)} \dim(V^{S_\lambda}) m_\lambda \in \Lambda\,.$$

Here $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_l} \subset S_n$ is the Young subgroup, and V^{S_λ} are the invariants. Note that both S_α and m_α are also defined when $\alpha = (\alpha_1, ..., \alpha_l)$ is only a strict composition of n, in other words ones of the 2^{n-1} finite sequences of positive integers summing to n, so that the sorted ones are just the partitions.

If V is a representation of a product of k factors of the symmetric group, we will denote the Frobenius character by

$$\mathcal{F}_{X_1,\ldots,X_k} V \in \mathbb{C}[x_{i,j}]^{S_n \times \cdots \times S_n},$$

which is a function in k sets of variables, $X_i = (x_{i,1}, x_{i,2}, ...)$, individually symmetric in each one. For modules with one or more gradings, which in this paper are bounded below in the negative degree, the Frobenius character will encode the degree in some generating variables $q_1, ..., q_k$. For instance, for bigraded modules which appear in this paper we will use the variables q, t, and so we will have

$$\mathcal{F}V = \sum_{i,j} q^i t^j \mathcal{F}V^{(i,j)} \in \Lambda_{\mathbb{Z}((q,t))}$$

where $V^{(i,j)}$ is the homogeneous component of the bigrading. For finitely generated modules, this can also be considered as an element of $\Lambda_{q,t} = \Lambda_{\mathbb{Q}(q,t)}$ by summing the generating function.

Suppose $V = M$ is a graded module over a polynomial ring $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, ..., x_n]$, using bold letters for sets of variables, together with an action of S_n which intertwines the action on the variables $\sigma(x_i) = x_{\sigma_i}$. In other words, M is a graded module over the smash product $\mathbb{C}[\mathbf{x}] \ltimes S_n$. Then we have the Frobenius character $\mathcal{F}M \in \Lambda_q$ regarding M as a vector space. In this case, the Frobenius character of the Euler characteristic is determined by plethystic substitution. In other words,

$$\sum_{i \geqslant 0} (-1)^i \mathcal{F} \operatorname{Tor}_i^{\mathbb{C}[\mathbf{x}]}(M, \mathbb{C}) = \mathcal{F}M\left[X(1-q)\right] \tag{4}$$

where $\mathbb{C} = \mathbb{C}[\mathbf{x}]/(x_1, ..., x_n)$ is the module on which x_i acts by zero, the Frobenius character is taken with respect to the induced action of S_n on the Tor groups, and for any symmetric function F we have that $F[Z(1-q)]$ is the image of F under the homomorphism defined on the power sum generators by

$$p_i \mapsto p_i(z_1, z_2, ...)(1-q^i), \quad Z = z_1 + z_2 + \cdots .$$

Note that there is no relation between the symmetric function variables labeled by z_i and the generators of the ring. We hope this does not create confusion when the symmetric function variables are given by the alphabet X.

We also have the plethystic substitution $F[-X]$ which is the result of substituting $p_i(Z) \mapsto -p_i(Z) = (-1)^i \omega p_i(Z)$, where ω is the Weyl involution.

3.2 The nabla operator

Let $\Lambda_{q,t}$ be the ring of symmetric over $\mathbb{C}(q,t)$. Let $\tilde{H}_\lambda = \tilde{H}_\lambda(X;q,t)$ denote the modified Macdonald polynomial

$$\tilde{H}_\lambda(X;q,t) = t^{n(\lambda)} J_\lambda[X/(1-t^{-1}); q, t^{-1}],$$

and let ∇ be the operator of Bergeron-Garsia-Haiman-Tesler [BGHT99] defined by

$$\nabla \tilde{H}_\lambda(X;q,t) = q^{n'(\lambda)} t^{n(\lambda)} \tilde{H}_\lambda(X;q,t), \tag{5}$$

where

$$n(\lambda) = \sum_i (i-1)\lambda_i, \quad n'(\lambda) = n(\lambda') = \sum_i \binom{\lambda_i}{2} \tag{6}$$

as defined in [Mac95]. Here λ' is the transposed partition, and notice that $n'(\alpha)$ is defined for compositions α, not just partitions. When there is more than one symmetric function alphabet present, we will suppose that $\nabla = \nabla_X$ acts on the symmetric functions in the X variables.

Then the nabla positivity conjecture of Bergeron, Garsia, Haiman, Tesler is as follows:

Conjecture 3.1 ([BGHT99], Conjecture I)**.** For any partitions λ, μ of norm n, and $k \geqslant 1$, we have that

$$(-1)^{\iota(\lambda)} \left(\nabla^k s_{\lambda'}, s_\mu\right) \in \mathbb{Z}_{\geqslant 0}[q,t], \tag{7}$$

where the inner product is the Hall inner product, and

$$\iota(\lambda) = \binom{l(\lambda)}{2} - \sum_{\lambda_i < i-1} (i-1-\lambda_i) = \sum_i \min(\lambda_i, i-1)$$

is the number of cells of λ below the main diagonal.

3.3 Standardization and super labels

Fix n corresponding to degree, and define a label to be an n-tuple of positive integers $\mathbf{a} = (a_1, ..., a_n)$ with $a_i \geqslant 1$. We define a (strict) composition of n by $\alpha(\mathbf{a}) = (\alpha_1, ..., \alpha_l)$, where $\alpha_i = \#\{j : a_j = c_i\}$, and $c_1 < \cdots < c_l$ are the numbers that appear at least once in $\mathbf{a}$. For instance, $\alpha(2, 2, 4, 1, 4, 2, 1, 5, 4) = (2, 3, 3, 1)$. We will set $X_{\mathbf{a}} = x_{a_1} \cdots x_{a_n}$. We define the q-automorphism factor

$$\mathrm{aut}_q(\mathbf{a}) = \mathrm{aut}_q(\alpha(\mathbf{a})), \quad \mathrm{aut}_q(\alpha) = \prod_i [\alpha_i]_q!$$

Let $\mathbf{m} \in \mathbb{Z}^n_{\geqslant 0}$, and write $|\mathbf{m}| = m_1 + \cdots + m_n$. We will be interested in $(d+1)$-tuples $(\mathbf{m}, \mathbf{a}_1, ..., \mathbf{a}_d)$ where each $\mathbf{a}_i$ is a label. We define the class $[\mathbf{m}, \mathbf{a}_1, ..., \mathbf{a}_d]$ to be the orbit of the diagonal action of S_n by simultaneously permuting the entries. We will identify these orbits with their minimal representatives, for which $\mathbf{m}$ is in decreasing order, and each $\mathbf{a}_i$ is in increasing order when there is a tie, lexicographically. Such tuples are called *sorted*. We similarly have $\alpha = \alpha(\mathbf{m}, \mathbf{a}_1, ..., \mathbf{a}_d)$ defined so that S_α is the stabilizer subgroup for the diagonal action of S_n, and similarly for $\mathrm{aut}_q(\mathbf{m}, \mathbf{a}_1, ..., \mathbf{a}_d)$. For instance, the sorted representative of

$$[\mathbf{m}, \mathbf{a}, \mathbf{b}] = [(2, 1, 2, 2, 0, 2, 0), (2, 2, 2, 2, 3, 1, 3), (1, 3, 1, 1, 1, 2, 1)]$$

is $((2, 2, 2, 2, 1, 0, 0), (1, 2, 2, 2, 2, 3, 3), (2, 1, 1, 1, 3, 1, 1))$. Then we have $\alpha(\mathbf{m}, \mathbf{a}, \mathbf{b}) = (1, 3, 1, 2)$, and

$$\mathrm{aut}_q(\mathbf{m}, \mathbf{a}, \mathbf{b}) = [1]_q![3]_q![1]_q![2]_q! = (1 + q)^2(1 + q + q^2).$$

We briefly recall some facts from [HHL$^+$05], Section 4. First, we have the super alphabet

$$\mathcal{A} = \mathbb{Z}_+ \cup \mathbb{Z}_- = \{1, \bar{1}, 2, \bar{2}, ...\}.$$

The integers i are called positive, while the overlined numbers $\bar{i}$ are called negative. The set is totally ordered by $\{1 < \bar{1} < 2 < \bar{2} < \cdots\}$, which is referred to there as $<_1$. Let the negative of a letter be the result of reversing the sign, and let absolute value remove any sign leaving only the positive part. In other words, $-c = \bar{c}$, $-\bar{c} = c$, and $|c| = |\bar{c}| = c$ for $c \in \mathbb{Z}_{\geqslant 1}$. We set $X_{\bar{a}} = X_a$.

We will make use of the following definition

Definition 3.2. The standardization $\sigma = \mathrm{Std}(\mathbf{a})$ of a label $\mathbf{a} \in \mathcal{A}^n$ is the unique permutation σ such that $\mathbf{a}_{\sigma^{-1}}$ is weakly increasing, and the restriction of σ to $\mathbf{a}^{-1}(\{x\})$ is increasing if x is positive, decreasing if x is negative. Here we are viewing $\mathbf{a}$ as a function $\{1, ..., n\} \to \mathcal{A}$.

For instance, $\mathrm{Std}((2, \bar{1}, 1, 4, 2, \bar{1}, \bar{1})) = (5, 4, 1, 7, 6, 3, 2)$.

3.4 Previous results

We recall the results from [CM20], which we refer to for more details.

We begin by recalling the dinv statistic.

Definition 3.3. Let $\mathbf{m} \in \mathbb{Z}^n_{\geqslant 0}$, let $\mathbf{a}, \mathbf{b}$ be labels, let $k \geqslant 1$, and suppose that $(\mathbf{m}, \mathbf{a})$ are sorted. We define

$$\mathrm{dinv}_k(\mathbf{m}, \mathbf{a}, \mathbf{b}) = \sum_{i<j} \mathrm{dinv}_k^{i,j}(\mathbf{m}, \mathbf{a}, \mathbf{b}) \tag{8a}$$

where

$$\mathrm{dinv}_k^{i,j}(\mathbf{m}, \mathbf{a}, \mathbf{b}) = \max\left(m_j - m_i - 1 + k + \epsilon(a_i, a_j) + \epsilon(b_i, b_j), 0\right), \tag{8b}$$

and $\epsilon(a_1, a_2)$ is one if $a_1 > a_2$, zero otherwise.

For just one label $\mathbf{a}$, we define $\mathrm{dinv}_k(\mathbf{m}, \mathbf{a})$ as the result of removing $\epsilon(b_i, b_j)$ from (8b), which has the same effect as setting $\mathbf{b} = (1^n)$.

Recall that an $n \times n$ Dyck path π is a path in the $n \times n$ grid starting at $(0, 0)$ and ending at (n, n), traveling only North and East, and never crossing below the main diagonal [Hag08]. It will often be denoted with 1's signifying North steps and 0's for East steps, so that the path in Figure 1 would be given by $\pi = 111001101000$. A Dyck path is determined uniquely by the set

$$D(\pi) = \{(i, j) : 1 \leqslant i < j \leqslant n \text{ is between the path and the diagonal}\}.$$

We will define a partial Dyck path [CM18] as a pair (π, l) where l is an most the number of trailing East steps in π. We will write them by simply leaving off that many zeros from the end of π, in other words write 1110011010 instead of $(\pi, 2)$ for the above Dyck path π.

It is not hard to show that the following definition defines a Dyck path:

Definition 3.4. Fix $k \geqslant 0$, suppose $(\mathbf{m}, \mathbf{a})$ is sorted, and let $i < j$. We will say that i attacks j if

$$m_j - m_i - 1 + k + \epsilon(a_i, a_j) \geqslant 0.$$

Let $\pi = \pi_k(\mathbf{m}, \mathbf{a})$ denote the Dyck path such that the elements of $D(\pi)$, are the pairs $i < j$ for which i attacks j.

We now have that

$$\mathrm{dinv}_k(\mathbf{m}, \mathbf{a}, \mathbf{b}) = \mathrm{dinv}_k(\mathbf{m}, \mathbf{a}) + \mathrm{inv}_{\pi_k(\mathbf{m},\mathbf{a})}(\mathbf{b}) \tag{9}$$

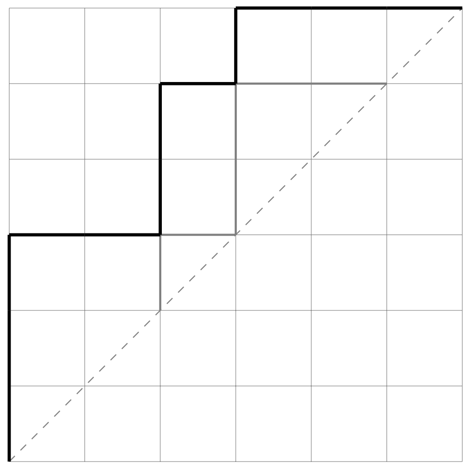

Figure 1: The Dyck path $\pi = 111001101000$. The boxes between the path and the diagonal are $D(\pi) = \{(1,2),(1,3),(2,3),(3,4),(3,5),(4,5),(4,6),(5,6)\}$, and we have $\alpha(\pi) = (2,1,2,1)$.

where

$$\mathrm{inv}_\pi(\mathbf{b}) = \#\{(i,j) \in D(\pi) : b_i > b_j\}. \tag{10}$$

We can also determine the composition $\alpha(\mathbf{m},\mathbf{a})$ from $\pi_0(\mathbf{m},\mathbf{a})$. Let $\alpha = \alpha(\pi)$ be defined by drawing lines through all edges, and recording the points where they contact the diagonal. In other words, α is the composition whose endpoints are the result of intersecting the vertical and horizontal lines with the diagonal. Then it is can be checked that $\alpha(\pi_0(\mathbf{m},\mathbf{a})) = \alpha(\mathbf{m},\mathbf{a})$. For arbitrary $k \geqslant 0$, the composition $\alpha(\mathbf{m},\mathbf{a})$ is a refinement of $\alpha(\pi_k(\mathbf{m},\mathbf{a}))$.

We have an LLT polynomial

$$\xi_\pi[Y;q] = \sum_{\mathbf{b}} q^{\mathrm{inv}_\pi(\mathbf{b})} Y_{\mathbf{b}}. \tag{11}$$

As we explain below, this polynomial appears as the Frobenius character of the equivariant cohomology of the regular semisimple Hessenberg variety associated to π, and is the plethystically transformed chromatic symmetric function of Stanley, which was proved in [BC15, SW12]. It is also the LLT polynomial that appears in [CM18].

The main theorem of [CM20] states

Theorem 3.5. *For any $k \geqslant 1$, we have*

$$\nabla_X^k e_n\left[\frac{XY}{(1-q)(1-t)}\right] = \sum_{[\mathbf{m},\mathbf{a},\mathbf{b}]} \frac{t^{|\mathbf{m}|} q^{\mathrm{dinv}_k(\mathbf{m},\mathbf{a},\mathbf{b})}}{(1-q)^n \mathrm{aut}_q(\mathbf{m},\mathbf{a},\mathbf{b})} X_{\mathbf{a}} Y_{\mathbf{b}}$$

$$= \sum_{[\mathbf{m},\mathbf{a}]} \frac{t^{|\mathbf{m}|} q^{\mathrm{dinv}_k(\mathbf{m},\mathbf{a})}}{(1-q)^n \mathrm{aut}_q(\mathbf{m},\mathbf{a})} X_{\mathbf{a}} \xi_{\pi_k(\mathbf{m},\mathbf{a})}[Y;q]. \tag{12}$$

We extend Definition 3.3 to super alphabets by setting

$$\epsilon(b_1,b_2) = \begin{cases} 1 & b_1 >_1 b_2 \text{ or } b_1 = b_2 \text{ is negative, or} \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

We similarly replace

$$\mathrm{inv}_\pi(\mathbf{b}) = \sum_{(i,j)\in D(\pi)} \epsilon(b_i,b_j).$$

Notice that $\pi_k(\mathbf{m},\mathbf{a})$ makes sense for super alphabets as well. Then

Corollary 3.6. *Theorem 3.5 holds for super alphabets, substituting* (13) *into* dinv_k *for negative labels.*

Proof. The result holds for super labels in $\mathbf{b}$ using equation (12), and the desired statement about $\xi_\pi[Y;q]$. We then apply the symmetry between the X and Y variables.

□

Example 3.7. Let $\pi = 110100$. Then we have

$$\xi_\pi = m_3 + (2\,q+1)\,m_{2,1} + \left(q^2+4\,q+1\right) m_{1,1,1}.$$

We can see that the coefficient of $m_{2,1}$ in $\omega\xi_{110100}$ is

$$q^{\mathrm{dinv}_\pi((\bar{1},\bar{1},\bar{2}))} + q^{\mathrm{dinv}_\pi((\bar{1},\bar{2},\bar{1}))} + q^{\mathrm{dinv}_\pi((\bar{2},\bar{1},\bar{1}))} = q^2 + 2q.$$

3.5 Affine permutations

We have the group of (extended) affine permutations

$$W = \{w : \mathbb{Z} \to \mathbb{Z} : w_{i+n} = w_i + n\}. \tag{14}$$

Each such permutation is uniquely determined by its values on the elements $\{1,...,n\}$, and we will often describe it elements in window notation $w = (w_1,...,w_n)$. We have a homomorphism $W \to \mathbb{Z}$ given by $w \mapsto d$, where d is the unique number satisfying $w_1+\cdots+w_n = dn(n+1)/2$, whose kernel W_0 is the usual group of affine permutations, which is the affine Weyl group in the SL_n case. We then have the decomposition into W_0 orbits

$$W = \bigsqcup_{d \in \mathbb{Z}} W_d, \quad W_d = W_0 \rho^d = \rho^d W_0,$$

where $\rho \in W$ is the rotation element $\rho_i = i + 1$.

The Bruhat order on W is defined as the usual order on W_0 which is generated by reflections, and so that elements in different W_d components are incomparable, and $v\rho^d \leqslant w\rho^d$ if and only if $v \leqslant w$. We have the usual inversion statistic

$$\mathrm{inv}(w) = \sum_{1 \leqslant i < j \leqslant n} \mathrm{inv}^{i,j}(w), \quad \mathrm{inv}^{i,j} = \max\left(\lceil (w_i - w_j)/n \rceil, \lceil (w_j - w_i)/n - 1 \rceil\right),$$

which is independent of ρ, $\mathrm{inv}(w\rho^d) = \mathrm{inv}(\rho^d w) = \mathrm{inv}(w)$.

If $J \subset \{0, ..., n-1\}$ is nonempty, we have the finite subgroups $W_J \subset W$ generated by s_i for $i \notin J$. The subsets $J \subset \{1, ..., n-1\}$ are in bijection with compositions $\alpha = (\alpha_1, ..., \alpha_l)$ of n in such a way that $j \in J$ if $j, j+1$ are in different blocks of α, and in this case $W_J = S_\alpha$ is just the Young subgroup of $S_n \subset W$. The other W_J are conjugates of the Young subgroups by the rotation element ρ. We have a Bruhat order on arbitrary double cosets $W_J \backslash W / W_{J'}$, which is the same as the order induced by the map of posets $W_J \backslash W / W_{J'} \subset W$ which selects the unique minimal representative w_- of each coset, or alternatively the unique maximal one w_+, which will be denoted $w_\pm \in W_J \backslash W / W_{J'}$.

We now explain a correspondence between affine permutations and the combinatorial statistics of the last section. Let W_+ denote the set of *positive* affine permutations

$$W_+ = \{w \in W : w(\mathbb{Z}_{\geqslant 1}) \subset \mathbb{Z}_{\geqslant 1}\}, \tag{15}$$

in other words those affine permutations whose window notation contains only positive integers. In [CM20], we considered a bijection

$$\mathrm{aff} : \{[\mathbf{m}, \mathbf{a}, \mathbf{b}] : X_\mathbf{a} Y_\mathbf{b} = X^\alpha Y^\beta\} \longleftrightarrow S_\beta \backslash W_+ / S_\alpha.$$

for each pair of compositions α, β. It is defined by $\mathrm{aff}(\mathbf{m}, \mathbf{a}, \mathbf{b}) = S_\beta \, \mathrm{aff}_0(\mathbf{m}, \mathbf{a}, \mathbf{b}) S_\alpha$, where

$$\mathrm{aff}_0(\mathbf{m}, \mathbf{a}, \mathbf{b}) = \mathrm{Std}_>(\mathrm{rev}(\mathbf{b})) t(\mathbf{m}) \, \mathrm{Std}_<(\mathbf{a})^{-1} \tag{16}$$

is a maximal representative of the double coset. Here $t(\mathbf{m}) = (n + m_1 n, ..., 1 + m_n n)$ is the maximal representative of its coset in $S_n \backslash W_+ / S_n$, and $\mathrm{rev}(\mathbf{b})$ is the result of writing $\mathbf{b}$ in the reverse order. If no $\mathbf{b}$ label is specified, we will define $\mathrm{aff}(\mathbf{m}, \mathbf{a}) = \mathrm{aff}(\mathbf{m}, \mathbf{a}, (1^n)) \in S_n \backslash W_+ / S_\alpha$.

We can now translate some of the constructions from the last section in terms of the corresponding double coset under aff. First, we can recover the dinv statistic by $\mathrm{dinv}_k(\mathbf{m}, \mathbf{a}, \mathbf{b}) = \mathrm{dinv}_k(w_+)$ where $w_+ = \mathrm{aff}_0(\mathbf{m}, \mathbf{a}, \mathbf{b})$, and

$$\mathrm{dinv}_k(w) = \sum_{1 \leqslant i < j \leqslant n} \max\left(k - \mathrm{inv}^{i,j}(w^{-1}), 0\right) = k\binom{n}{2} - \sum_{1 \leqslant i < j \leqslant n} \mathrm{inv}_k^{i,j}(w^{-1}), \quad (17)$$

where $\mathrm{inv}_k^{i,j}(w^{-1}) = \min(k, \mathrm{inv}^{i,j}(w^{-1}))$, which also holds if there is no third label, $\mathbf{b} = (1^n)$. We also have a rule

$$\mathrm{dinv}_k(\mathbf{m}, -\mathbf{a}, \mathbf{b}) = \mathrm{dinv}_k(w) \quad (18)$$

where w is the minimal element in the maximal right coset $w_+ S_\alpha \subset \mathrm{aff}(\mathbf{m}, \mathbf{a}, \mathbf{b})$, for $w_+ = \mathrm{aff}_0(\mathbf{m}, \mathbf{a}, \mathbf{b})$. The composition $\alpha' = \alpha(\mathbf{m}, \mathbf{a})$ is the unique one with the property that $S_{\alpha'}$ is the stabilizer of S_n acting on $w_+ S_\alpha$. Finally, the Dyck path $\pi = \pi_k(\mathbf{m}, -\mathbf{a})$ is the one determined by

$$D(\pi) = \left\{(i, j) : \mathrm{inv}^{i,j}(w_-^{-1}) < k\right\} \quad (19)$$

where $i < j$ are in different blocks of α', where w_- is the minimal element of $\mathrm{aff}(\mathbf{m}, \mathbf{a})$ in $S_n \backslash W_+ / S_\alpha$. It will be denoted $\pi_k(S_n w S_\alpha)$.

4 GKM spaces

We review some necessary background on equivariant Borel-Moore homology and GKM spaces. We begin wih the general setup, and then present the relevant examples.

4.1 General setup

Let X be a (possibly singular) complex projective variety variety together with an action of an algebraic torus $T = (\mathbb{C}^*)^d$. Suppose that X satisfies the Goresky-Kottwitz-Macpherson (GKM) conditions, namely that the fixed point set X^T is finite, there are finitely many one-dimensional orbits, and X is equivariantly formal. The theorem of Goresky-Kottwitz-Macpherson [GKM97] says that the equivariant cohomology $H_T^*(X)$, which in this paper has complex coefficients, injects into the cohomology of the fixed point set $H_T^*(X) \hookrightarrow H_T^*(X^T)$, and that the image can be described by algebraic conditions determined by the weighted moment graph.

Let $H^T_*(X)$ be the equivariant Borel-Moore homology, as defined in [Bri98, EG96]. Then as in these references, we have that $H^T_*(X)$ is free over $H^*_T(pt) = \mathbb{C}[t_1, ..., t_d]$ which will be denoted by S, and the localization map $i_* : H^T_*(X^T) \to H^T_*(X)$ becomes an isomorphism after inverting finitely many characters of T. The homology may then be identified as an S-submodule of the localization

$$H^T_*(X^T) \otimes_S F = F \cdot X^T = \bigoplus_{p \in X^T} F \cdot p$$

via i_*^{-1}, where $F = \mathbb{C}(t_1, ..., t_d)$ is the field of fractions of S. Note that the homological grading is in the negative direction, so that multiplication by S action raises degree. Suppose that X is paved by T-invariant affines, meaning that we have $X = \bigsqcup_i U_i$, where $U_i \cong \mathbb{A}^{d_i}$ for some d_i, and

$$X_i = \overline{U}_i \subset \bigcup_{j \leqslant i} U_j \tag{20}$$

for some partial order on the indices. Then by [Gra01] Proposition 2.1, the fundamental classes $A_i = [X_i]_T$ freely generate $H^T_*(X)$ as a module over $S = H^T_*(pt)$. By the above assumptions about X and the same proposition, there are dual generators $\xi_i \in H^*_T(X)$ satisfying $\xi_i(A_j) = \delta_{i,j}$.

Let $G = (V, E, \tau)$ be the moment graph of X, where $V = X^T$ is the fixed point set, E is the set of one-dimensional orbits, and $\tau : E \to S$ is the weight function of the action. In general, the edge set can directed in more than one way by choosing a sufficiently generic one-dimensional subtorus $T' \subset T$, and deciding that the edge points towards the South pole of the corresponding one-dimensional orbit, where the South pole is determined as the attracting point with respect to T', see [GHZ06, Tym05, Tym08, GZ03]. This determines a potentially different partial order on V. For instance, while the Bruhat order is the standard one to use for flag varieties, one may also consider the conjugates of this order by permutations, which will be an important observation in Section 5.

The homological and cohomological generators then satisfy

$$A_p = \sum_{q \leqslant p} c_{p,q} q, \quad \xi_p = \sum_{q \geqslant p} d_{p,q} q \tag{21}$$

as elements of $F \cdot V$. Moreover, the leading terms are given by

$$c_{p,p} = d_{p,p}^{-1}, \quad d_{p,p} = \mathrm{LT}(p) = \prod_{e \in E^+(p)} \tau(e),$$

where $E^+(p) = \{(p,q) : q \in N^+(p)\}$ is the set of outgoing edges, and $N^+(p)$ is the set of endpoints. The leading term follows from combining the GKM relations with the triangularity, and in some cases the other coefficients can be deduced using the type of argument used by Knutson and Tao in [KT01]. The classes ξ_p are often referred to as canonical classes, and if some conditions known as the Palais-Smale conditions on the GKM data are satisfied, they are unique [GZ03]. One condition that forces the uniqueness is if the degree strictly increases along increasing chains in the poset determined by (V, E), which is true in the case of the affine flag variety, but not the regular semisimple Hessenberg variety or the unramified affine Springer fiber Y_k defined below.

4.2 The affine flag variety

We recall the construction of the affine flag variety, referring to [Kum02, LSS10, LLM$^+$14] for more details.

Let $\mathcal{K} = \mathbb{C}((z))$ and $\mathcal{O} = \mathbb{C}[[z]]$, and consider the groups $G(\mathcal{O}) \subset G(\mathcal{K})$ for $G = GL_n$. We have the affine Weyl group, which is the set of affine permutations from (14). It acts on $\mathcal{K}^n$ by $w \cdot e_j = e_{w_j}$ for $j \in \mathbb{Z}$, where $e_{i+dn} = z^{-d} e_i \in \mathcal{K}^n$ for $i \in \{1, ..., n\}$, identifying it as a subgroup $W \subset G(\mathcal{K})$. If α is a composition, let $P_\alpha \subset GL_n$ denote the corresponding parabolic subgroup, so that $P_{(1,...,1)} = B$, the upper triangular matrices. If $J \subset \{0, ..., n-1\}$ is nonempty as in Section 3.5, we have the parahoric subgroup $I_J \subset G(\mathcal{K})$ containing the group W_J, so that $I_{\{0,...,n-1\}}$ is the usual Iwahori subgroup, and $I_{\{0\}} = G(\mathcal{O})$. If $J \subset \{1, ..., n-1\}$ corresponds to the composition α, then I_J is the inverse image of P_α under the map $\pi : G(\mathcal{O}) \to G$ which evaluates at $z = 0$. The others may be obtained by conjugating these subgroups by ρ.

Let $Y^J = G(\mathcal{K})/I_J$ denote the affine flag variety. If no parabolic is specified, let $Y = Y^{\{0,...,n-1\}}$ denote the one corresponding to the standard Iwahori subgroup, i.e. the full flag variety, and denote $X = Y^{\{0\}}$ for the affine Grassmannian. Then parametrically we have that $X = \{V \subset \mathcal{K}^n\}$ where V is an $\mathcal{O}$-submodule, and there exists $N > 0$ so that $z^{-N}\mathcal{O} \subset V \subset z^N \mathcal{O}$. The full flag variety may be described by

$$Y = \{V_0 \subset \cdots \subset V_n : V_0 = zV_n\},$$

where each $V_i \in X$, and $\dim(V_{i+1}/V_i) = 1$. We may also consider a flag as periodic and infinite by setting $V_i = zV_{i+n}$, and so refer to V_i for any $i \in \mathbb{Z}$. The image of a Weyl group element $w \in W$ determines a flag by $V_i = \langle e_{w_j} : j \leqslant i \rangle$ where e_i was defined above. Each Y^J decomposes as a disjoint union

$$Y^J = \bigsqcup_{d \in \mathbb{Z}} Y_{SL_n}^{\rho^d(J)},$$

where $Y_{SL_n}^J$ is the partial affine flag variety for SL_n. Each $\rho^d(J)$ is obtained by applying ρ to the elements of J modulo n, and the Springer action is conjugated by ρ in the compatible way.

We now describe the equivariant homology and cohomology of Y^J. There is an action on Y^J by the extended torus $\tilde{T} \cong (\mathbb{C}^*)^n$, which is the small torus $T \subset GL_n$ together with loop rotation. The ground ring, which is the equivariant cohomology of a point with respect to $T, \tilde{T}$ are given by $S = \mathbb{C}[x_1, ..., x_n]$ and $\tilde{S} = \mathbb{C}[x_1, ..., x_n, \epsilon]$ respectively, where $x_1, ..., x_n$ are the weights of T, and ϵ is the parameter corresponding to loop rotation. We have the root elements

$$\alpha_i = \alpha_{i,i+1}, \quad \alpha_{i,j} = \lambda_i - \lambda_j, \quad \lambda_i = x_{\bar{i}} + \frac{i - \bar{i}}{n}\epsilon$$

where $\bar{i}$ is the unique element of $\{1, ..., n\}$ which is congruent to i modulo n. Then W acts by automorphisms of $\tilde{S}$ in such a way that $w(\lambda_i) = \lambda_{w_i}$.

The fixed points of both $T, \tilde{T}$ are given by $V = W/W_J$ via the inclusion described above, and the corresponding point will be labeled p_w. The edges in the GKM graph are determined by

$$N^+(w) = \{v \leqslant_{bru} w : w = t_{i,j} v\}. \tag{22}$$

for some affine transposition $t_{i,j}$, which is an element of W that switches two elements $i, j \in \mathbb{Z}$ which are not congruent modulo n, and is necessarily in W_0. We have the corresponding definition v, w are replaced by cosets vW_J, wW_J, and in fact is also the same as replacing the cosets by minimal representatives in the Bruhat order. The weight function is determined by

$$\tau(t_{i,j} w, w) = \alpha_{i,j}, \tag{23}$$

whose degree is the affine inversion number $\mathrm{inv}(w)$ defined in Section 3.5.

The cells in the affine paving (20) are the Schubert cells $\Omega_w^J = IwI_J$. The Schubert varieties are the closures

$$V_w = \overline{\Omega}_w = \bigsqcup_{v \leqslant w} \Omega_v.$$

The homology is defined by treating Y^J as an ind-variety using the Bruhat filtration, see [Kum02]. If $P \subset W/W_J$ is a finite lower set, meaning a finite subset for which

$$v \leqslant w, \ w \in P \Rightarrow v \in P,$$

then the subspace

$$V_P^J = \bigcup_{uW_J \in P} V_{uW_J}^J = \bigsqcup_{uW_J \in P} \Omega_{uW_J}^J$$

is a closed subvariety of Y^J which is GKM with respect to the action of $\tilde{T}$. We then have the direct limit

$$H_*^{\tilde{T}}(Y^J) = \lim_{\to} H_*^{\tilde{T}}\left(V_P^J\right) \tag{24}$$

over all lower sets $P \subset W/W_J$.

We have commuting left and right actions called *dot* and *star* of W on $H_*^{\tilde{T}}(Y)$ induced by the action by multiplication on $F \cdot W$ via i_*^{-1}, in which the dot action acts on the ground field $\tilde{F}$ by the automorphisms described above. Moreover, we can recover the homology of both the GL_n version Y and the parabolics Y^J from the SL_n case Y_{SL_n}, via the formula

$$H_*^{\tilde{T}}(Y^J) = \bigoplus_d \rho^d H_*^{\tilde{T}}(Y_{SL_n}^J), \quad H_*^{\tilde{T}}(Y^J) \cong H_*^{\tilde{T}}(Y)^{W_J}, \tag{25}$$

the second isomorphism being one of the fundamental properties of the Springer action. Here M^{W_J} are the invariants with respect to $W_J \subset W_0$ acting on the right by the star action twisted by the star action, which is defined by identifying $W_J = \rho^k S_\alpha \rho^{-k}$ for some k. The inverse of the corresponding generators $A_{wS_\alpha}^J$ are the result of symmetrizing A_w by W_J, when w is minimal in the coset W/W_J.

Though we will not need this definition, the cohomology is defined as the subset of the inverse limit

$$H_{\tilde{T}}^*(X) \subset \lim_{\leftarrow} H_{\tilde{T}}^*(X_w) \tag{26}$$

as graded $\tilde{S}$-modules [Gra01]. Equivalently, one may take cohomology to be all those elements ξ in the (non graded) inverse limit such that $\xi(A_w) = 0$ for all but finitely many w. For Y, this agrees with the first definition, and is more useful when the cohomology is defined algebraically as the dual module to the nil Hecke algebra [LLM+14], defined in Section 4.3 below. In either case, there are canonical elements $\xi_v \in H_T^*(Y)$ which restrict to ξ_v on each $H_T^*(V_w)$, and which are the Kostant-Kumar basis [KK86].

4.3 The nil Hecke ring

We now recall the nil Hecke algebra $\mathbb{A}_{af}$ of Kostant and Kumar, which is an algebraic presentation of $H_*^{\tilde{T}}(Y)$, for which we refer to [KK86, LLM+14]. Let $\tilde{F} = \mathbb{C}(\mathbf{x}, \epsilon)$ be the fraction field field of $\tilde{S}$. Then we have a left and a right action of W on the free $\tilde{F}$-vector space $\tilde{F} \cdot W$, where the left action intertwines the scalars by

$$wf(\lambda_1, ..., \lambda_n) = f(\lambda_{w_1}, ..., \lambda_{w_n})w,$$

and the weights λ_i are the ones defined above.

Now consider the action of the operators

$$A_i = \frac{1 - s_i}{\alpha_i} \in \mathrm{End}\left(\tilde{F} \cdot W\right),$$

for $i \in \{0, ..., n-1\}$, where $s_i \in W$ is the simple transposition. They define the action of Kostant and Kumar's nil Hecke ring $\mathbb{A}_{af}$ [KK86] in type A, which is the $\mathbb{C}$-algebra generated by $(A_0, ..., A_{n-1})$ and $\tilde{S}$, subject to the following relations:

$$\begin{aligned} A_i\lambda = (s_i \cdot \lambda)A_i + (\alpha_i^\vee, \lambda) \cdot 1, \\ A_iA_i = 0, \\ A_iA_jA_i \cdots = A_jA_iA_j \cdots, \text{ if } s_is_js_i \cdots = s_js_is_j \cdots. \end{aligned} \tag{27}$$

Here $\alpha_i^\vee$ is the dual root. For more details, see [LLM$^+$14] Chapter 3, Section 6.

The following summarizes the connection between $\mathbb{A}_{af}$ and the homology of the affine flag variety in type A:

Proposition 4.1. *If $w = s_{i_1} \cdots s_{i_l}\rho^d \in W$ is a reduced word decomposition, then we have that $\varphi(A_{i_1} \cdots A_{i_l}\rho^d) = A_w$ under the isomorphism $\varphi : \tilde{F} \cdot W \to \tilde{F} \cdot V$ which sends w to the fixed point p_w. In particular, we have an isomorphism*

$$H_*^{\tilde{T}}(Y) = \bigoplus_d \rho^d \mathbb{A}_{af}$$

of $\tilde{S}$ modules which intertwines the left and right action of W.

The result also holds with T in place of $\tilde{T}$, though in this case the flag variety is not GKM. Using (25), we may recover the homologies of $H_*^{\tilde{T}}(Y^J)$.

Example 4.2. A typical basis element

$$A_{(1,6,2)} = A_2A_1A_0\rho^1 \in \mathbb{A}_{af}\rho^1 \subset H_*^{\tilde{T}}(Y)$$

is given by

$$\frac{p_{(3,2,4)}}{(\epsilon+x_1-x_2)(x_1-x_3)(x_2-x_3)}+\frac{p_{(2,4,3)}}{(\epsilon+x_1-x_3)(x_1-x_2)(x_2-x_3)}-$$
$$\frac{p_{(2,3,4)}}{(\epsilon+x_1-x_3)(x_1-x_2)(x_2-x_3)}+\frac{p_{(1,6,2)}}{(\epsilon+x_3-x_2)(x_1-x_3)(x_2-x_3)}-$$
$$\frac{p_{(1,5,3)}}{(\epsilon+x_2-x_3)(x_1-x_2)(x_2-x_3)}+\frac{p_{(1,3,5)}}{(\epsilon+x_2-x_3)(x_1-x_2)(x_2-x_3)}-$$
$$\frac{p_{(1,2,6)}}{(\epsilon+x_3-x_2)(x_1-x_3)(x_2-x_3)}-\frac{p_{(3,4,2)}}{(-x_2+\epsilon+x_1)(x_1-x_3)(x_2-x_3)}.$$

Notice that $(1,6,2)$ is minimal in $W/W_{\{2\}}$, so that $A_{(1,6,2)}$ maps to a generator of $H_*^{\tilde{T}}(Y^{\{2\}}) = H_*^{\tilde{T}}(Y^{\{0\}})\rho$. On the other hand, it maps to zero in $H_*^{\tilde{T}}(Y^{\{0,1\}})$ for dimension reasons since $(1,6,2)$ is not minimal in $W/W_{\{0,1\}}$.

4.4 The unramified affine Springer fiber

The affine Springer fiber [LS91] associated to an element $\gamma \in \mathrm{Lie}(G(\mathcal{K}))$ is

$$Y_\gamma^J = \left\{ gI_J \in Y^J : g^{-1}\gamma g \in \mathrm{Lie}(I_J) \right\}.$$

We will be interested in the *unramified* affine Springer fiber $Y_k \subset Y$ of [GKM04] associated to the element $\gamma_k = \mathrm{diag}(a_1 z^k, ..., a_n z^k)$, where a_i are distinct complex numbers and $k \geqslant 1$. We will also write Y_k^α to denote the corresponding parabolics $J = I_\alpha$ as for the flag variety. Then the action of the big torus $\tilde{T}$ on Y preserves Y_k, and the fixed points are all of W. It was shown in [GKM03] that Y_k^J is paved by affine spaces given by the intersections $Y_k^J \cap \Omega_{vW_J}^J$. We have the closures $\overline{\Omega_{vW_J}^J \cap Y_k^J}$, which are strictly contained in the intersections $Y_k^J \cap V_{vW_J}^J$. As above, we will drop the parahoric indices J to denote the full flag version.

For a finite lower set $P \subset W/W_J$, the subvarieties $Y_k \cap V_P$ are GKM with respect to $\tilde{T}$, and we now describe the GKM graph, referring to [GKM04]. An explicit description in the Grassmannian case for GL_n was also explained in [Kiv20]. The set of vertices is given by $V = W$, the entire Weyl group. We have

$$N_k^+(vW_J) = \{uW_J \leqslant_{bru} vW_J : u_- = t_{a,b}v_- = v_-t_{i,j},\ |i-j| < kn\}. \tag{28}$$

where u_-, v_- are the minimal representatives. The statistic $|i-j|$ does not depend on the representatives $\{i,j\}$ or their order, and is called the height of the transposition. Then the GKM graph of Y_k is the one for which the outgoing neighbors of vW_J are given by $N_k^+(vW_J)$. Notice that we obtain the set of neighbors $N^+(vW_J)$ for the affine flag variety as $k \to \infty$. The leading terms $LT_k(v)$ is also determined by the weight function (23). Just as in (22), the parabolic versions are the same as the ones obtained by taking minimal representatives of the cosets uW_J, vW_J. We will similarly denote by $\mathrm{inv}_k(vW_J) = |N_k^+(vW_J)|$ the number of outgoing edges.

The homology is defined as before as the direct limit

$$H_*^{\tilde{T}}(Y_k^J) = \varinjlim H_*^{\tilde{T}}(V_P^J \cap Y_k^J), \tag{29}$$

over lower sets $P \subset W/W_J$, and we have the basis elements by $A_{k,vW_J}^J = [V_{vW_J}^J \cap Y_k^J] \in H_*^{\tilde{T}}(Y_k^J)$. The natural map $H_*^{\tilde{T}}(Y_k) \to H_*^{\tilde{T}}(Y)$ is injective, since they are both contained in the fixed point set, and is compatible with the left and right actions of W.

As in the case of the flag variety, we can obtain the parabolic case by taking invariants for the Springer action:

Proposition 4.3. *The composition*

$$H_*^{\tilde{T}}(Y_k)^{W_J} \to H_*^{\tilde{T}}(Y_k) \to H_*^{\tilde{T}}(Y_k^J) \tag{30}$$

is an isomorphism, where the leftmost term is the invariants with respect to the right (i.e. Springer, star) action of W_J.

Remark 4.4. Results of this sort in the non-affine case can be found, for instance in [BM83]. We give a direct proof using the explicit description in fixed points.

Proof. We have $H_*^{\tilde{T}}(Y_k) = H_*^{\tilde{T}}(Y_k)^{W_J} \oplus K$ where K is the kernel of the multiplication on the right by $\sum_{w \in W_J} w$.

Consider the map $\pi_J : H_*^{\tilde{T}}(Y_k) \to H_*^{\tilde{T}}(Y_k^J)$. On fixed points it is determined by the fact that it sends the fixed point corresponding to $w \in W_+$ for $w \in W$ to the fixed point corresponding to the class of w in W/W_J. Thus we see that $\pi_J|_K = 0$. Moreover, we see that the composition (30) becomes an isomorphism when tensored with $\tilde{F}$, so it is injective.

It remains to show that π_J is surjective. One way to see this is to observe that the projection induces an isomorphism between the cell corresponding to $w \in W/W_J$ and the cell corresponding to the minimal representative $w_- \in W$. This implies $\pi_J(A_{k,w_-}) = A_{k,w}^J$. Alternatively, we can observe that $\pi_J(A_{k,w_-})$ has the leading term of expected degree, and therefore $\pi_J(A_{k,w_-})$ for different w generate the same submodule of $H_*^{\tilde{T}}(Y_k^J) \otimes_{\tilde{S}} \tilde{F}$ as $A_{k,w}^J$.

□

Example 4.5. For instance, we have an element

$$A_{1,(1,6,2)} = \frac{p_{(3,4,2)}}{(x_1 - x_2 + \epsilon)(x_1 - x_3)} - \frac{p_{(3,2,4)}}{(x_1 - x_2 + \epsilon)(x_1 - x_3)} + \frac{p_{(1,2,6)}}{(x_3 - x_2 + \epsilon)(x_1 - x_3)} - \frac{p_{(1,6,2)}}{(x_3 - x_2 + \epsilon)(x_1 - x_3)}. \tag{31}$$

Since we have an injective map $H_*^{\tilde{T}}(Y_k) \to H_*^{\tilde{T}}(Y)$, we must have that $A_{1,(1,6,2)}$ expands positively in terms of the A_w generators, and indeed using $\mathbb{A}_{af}$ we may check that $A_{1,(1,6,2)} = (x_2 - x_3)A_{(1,6,2)} - A_{(1,5,3)}$.

Remark 4.6. The equivariant cohomology may be defined as in Section 4.2, though now the inverse limit of graded algebras does not produce something finitely generated, since there are infinitely many classes in each degree. This can be resolved using the alternative version, by considering the subalgebra of classes ξ_w which vanish on all but finitely many generators $A_{k,v}$. However, unlike in the case of the flag variety, this definition depends on the choice of generators $A_{k,v}$, whereas there are other potential candidates for the generators. For instance, using the intersection of closures $Y_k \cap V_w$ in place of the closures of the intersections would result in a different limit. Defining the cohomology in the right way is interesting for the problem of generalizing Schubert positivity, but this is a red herring for this paper, and so will not be defined. We will only need the equivariant cohomology in the usual sense for each $Y_k \cap V_P$ individually.

4.5 The regular semisimple Hessenberg variety

We recall the GKM construction for the regular semisimple Hessenberg variety. We will refer to [DMPS92, AHM17, Tym08] for details.

The regular semisimple Hessenberg variety is determined by a *Hessenberg function*, which in type A is equivalent to the data of a Dyck path π, which in turn defines the *unit interval order* $i <_\pi j$ if $i < j$ and $(i,j) \notin D(\pi)$. Let M_π be the set of $n \times n$ matrices $(a_{i,j})$ for which $a_{i,j} = 0$ whenever $i >_\pi j$. Let $P_\alpha \subset GL_n(\mathbb{C})$ be a parabolic subgroup for which the conjugation action preserves M_π, which for any π includes the upper triangular matrices B. Let $\gamma = \mathrm{diag}(a_1, ..., a_n)$ be the diagonal matrix with all distinct complex entries. The regular semisimple Hessenberg variety is the subvariety of the usual complex partial flag variety given by

$$\mathcal{H}_\pi^\alpha = \left\{ gP_\alpha : g^{-1}\gamma g \in M_\pi \right\} \subset \mathrm{Fl}_\alpha .$$

The composition $\alpha = \alpha(\pi)$ from Section 3.4 determines a composition for which $P_\alpha \subset GL_n(\mathbb{C})$ fixes M_π, and so we have a corresponding variety $\mathcal{H}_\pi^{\alpha'}$ whenever $P_{\alpha'} \subset$

P_α, in other words α' refines α. In the example from Figure 1, we would have

$$M_\pi = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix} \right\}, \quad P_\alpha = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \right\},$$

and that M_π is preserved by conjugation by P_α, where $\alpha = (2,1,2,1)$.

There is an action of the maximal n-dimensional torus $T \subset GL_n(\mathbb{C})$, and $\mathcal{H}_\pi^\alpha$ is GKM with respect to this action, with fixed points given by the entire symmetric group $V = S_n$. The edge set is determined by

$$N_\pi^+(wS_\alpha) = \{vS_\alpha \leqslant_{bru} wS_\alpha : v_- = w_- t_{i,j}, (i,j) \in D(\pi)\} \tag{32}$$

where v_-, w_- are minimal coset representatives as above. We will denote the basis of homology by $A_{\pi,\sigma} \in H_T^*(\mathcal{H}_\pi)$. The generators of $H_T^*(\mathcal{H}_\pi^\alpha)$ may again be determined as in (25).

Example 4.7. For instance, we would have

$$A_{110100,(3,1,2)} = \frac{1}{x_1 - x_3} p_{(3,1,2)} - \frac{1}{x_1 - x_3} p_{(1,3,2)} = (x_2 - x_3) A_{(3,1,2)} - A_{(2,1,3)}, \tag{33}$$

where A_σ are the generators of the usual flag variety $H_T^*(\mathrm{Fl}_n)$.

We have the dot action of S_n on the left on $H_*^T(\mathcal{H}_\pi^\alpha)$ and $H_T^*(\mathcal{H}_\pi)$ which is compatible with (36). It was conjectured by Shareshian and Wachs, and then proved in two separate papers [BC15, SW16] that

$$\mathcal{F}_Y H_T^*(\mathcal{H}_\pi) = (1-q)^{-n} \xi_\pi[Y;q] = \omega \mathcal{X}_\pi[Y(1-q)^{-1};q] \tag{34}$$

where $\xi_\pi[Y;q]$ was defined in (11), and

$$\mathcal{X}_\pi[Y;q] = \sum_{\mathbf{b}:(i,j)\in D(\pi) \Rightarrow b_i \neq b_j} q^{\mathrm{inv}_\pi(\mathbf{b})} Y_{\mathbf{b}} \tag{35}$$

is Stanley's chromatic symmetric function.

The equivariant/non-equivariant homology/cohomology are given as follows:

$$H^*(\mathcal{H}_\pi) = (1-q)^{-n} \xi_\pi[Y(1-q);q] = \omega \mathcal{X}_\pi[Y;q], \qquad H_*(\mathcal{H}_\pi) = \omega \mathcal{X}_\pi[Y;q^{-1}]$$

$$H_*^T(\mathcal{H}_\pi) = \omega \mathcal{X}_\pi[Y(1-q)^{-1}; q^{-1}] = (1-q)^{-n}\omega\xi_\pi[Y; q^{-1}].$$

Since $\mathcal{H}_\pi$ is smooth (though perhaps not connected) [DMPS92], we have an isomorphism of equivariant Borel-Moore homology with cohomology. For $\alpha = (1, ..., 1)$, it is given in the fixed point basis by

$$H_*^T(\mathcal{H}_\pi) \to H_T^{*+\#D(\pi)}, \quad p_\sigma \mapsto \left(\prod_{(i,j)\in D(\pi)} (x_{\sigma_i} - x_{\sigma_j})\right) p_\sigma. \tag{36}$$

The multiplication map may be written as $f \mapsto f\Delta_\pi(\mathbf{z})$ where

$$\Delta_\pi(\mathbf{z}) = \prod_{(i,j)\in D(\pi)} (z_i - z_j),$$

and z_i is multiplication by the Chern class $p_\sigma z_i = x_{\sigma_i} p_\sigma$, which is natural to think of as right multiplication. Then Poincaré duality (on each connected component) is manifested via the identities

$$\omega\xi_\pi[Y; q^{-1}] = q^{-D(\pi)}\xi_\pi[Y; q], \qquad \mathcal{X}_\pi[Y; q^{-1}] = q^{-D(\pi)}\mathcal{X}_\pi[Y; q].$$

If α is some composition so that conjugation by P_α preserves M_π, then the fibers of the map $\mathcal{H}_\pi \to \mathcal{H}_\pi^\alpha$ are isomorphic to the product of usual flag varieties $\mathrm{Fl}_{\alpha_1} \times \cdots \times \mathrm{Fl}_{\alpha_l}$, as the Hessenberg condition is trivial on the fibers. We therefore have that

$$\mathcal{F}_Y H_T^*(\mathcal{H}_\pi^\alpha) = \xi_{\pi,\alpha}[Y; q] := \frac{1}{(1-q)^n \mathrm{aut}_q(\alpha)} \xi_\pi[Y; q],$$

$$\mathcal{F}_Y H_*^T(\mathcal{H}_\pi^\alpha) = (-q)^{-n}\omega_Y \xi_{\pi,\alpha}[Y; q^{-1}] = \frac{q^{n'(\alpha)-D(\pi)}}{(1-q)^n \mathrm{aut}_q(\alpha)} \xi_\pi[Y; q], \tag{37}$$

where $n'(\alpha)$ was defined in (6).

4.6 The Hessenberg paving of the affine Springer fiber

We now describe an explicit presentation of the paving by affine bundles over the Hessenberg varieties defined in [GKM03]. We refer to that paper for all details in this section.

Let $P \subset W/S_\alpha$ be a finite lower set, and suppose that $P = Q \cup S_n w S_\alpha$ for another lower set Q, where the union is disjiont, so that the double coset is maximal within P. Then we have the complement

$$E_{S_n w S_\alpha}^\alpha = \Omega_{S_n w S_\alpha}^\alpha := \bigsqcup_{v \in S_n w S_\alpha} \Omega_{vS_\alpha}^\alpha = V_P^\alpha - V_Q^\alpha$$

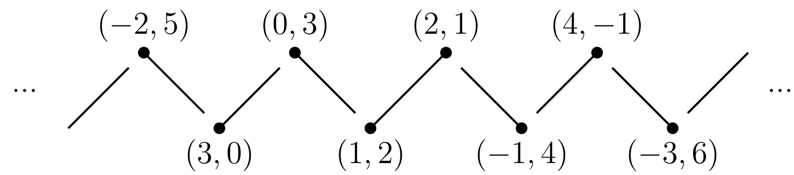

Figure 2: An illustration of the Hessenberg paving of Y_1 for $n = 2$. The fibers of the bundle $E_{S_2 w S_{1,1}}$ are shown using incomplete segments.

depends only on $S_n w S_\alpha$. Moreover, it is isomorphic to an affine bundle of rank $\mathrm{inv}(w_-)$ over the parabolic (non affine) flag variety $\mathrm{Fl}_{\alpha'} = GL_n/P_{\alpha'}$, where as above $w_+, w_- \in S_n w S_\alpha$ are the maximal and minimal coset representatives. The composition α' is the unique composition so that the parabolic subgroup $P_{\alpha'}$ is the stabilizer of GL_n acting on $w_- P_\alpha$, which is the same as the composition $\alpha' = \alpha(S_n w S_\alpha)$ from Section 3.5. Then we have that $E^\alpha_{S_n w S_\alpha} \subset V^\alpha_{w_+} = G(\mathcal{O}) w I_\alpha$, and the map $E^\alpha_{S_n w S_\alpha} \to \mathrm{Fl}_{\alpha'}$ can be expressed by

$$g(t) w I_\alpha \mapsto g(0) P_{\alpha'}, \quad g(t) \in \mathcal{O} = I_{(n)}.$$

By the results of [GKM03], we have that $Y^\alpha_k \cap E^\alpha_{S_n w S_\alpha}$ is an affine sub-bundle over the Hessenberg variety $\mathcal{H}^{\alpha'}_\pi \subset \mathrm{Fl}_{\alpha'}$, which form what is called a Hessenberg paving. The Dyck path is determined by $\pi = \pi_k(S_n w S_\alpha)$, which is the path from Section 3.5. It may be checked that $\alpha(\pi_k(S_n w S_\alpha))$ refines $\alpha' = \alpha(S_n w S_\alpha)$, so that the Hessenberg variety is well-defined. Geometrically, $\pi_k(S_n w S_\alpha)$ may be expressed as the unique path so that for $i < j$, we have

$$(i, j) \in D(\pi) \Leftrightarrow t_{a,b} w_- \in N^+_k(w_-),$$

by comparing with (19).

Example 4.8. Consider the case of $n = 2, k = 1$ shown in Figure 2. We see that $\pi_1(S_2(1,2)S_{1,1}) = 1100$ so that $\mathcal{H}_\pi = \mathrm{Fl}_2$, and that the bundle has rank zero, giving a copy of $\mathbb{CP}^1$ connecting $(1,2)$ to $(2,1)$. On the other hand, all other Dyck paths such as $\pi_1(S_2(0,3)S_{1,1})$ are equal to 1010, for which the Hessenberg variety consists of two points, in this case $\{(0,3),(-1,4)\}$. The bundle $E_{\{(0,3),(-1,4)\}} \cap Y_1$ has rank one, shown by incomplete line segments.

We may describe the equivariant Euler class of the fiber of $E^\alpha_{S_n w S_\alpha}$ over a fixed point:

$$e\left(E^\alpha_{S_n w S_\alpha}\big|_{\sigma w_- S_\alpha}\right) = \frac{\mathrm{LT}_k(\sigma w_- S_\alpha)}{\mathrm{LT}_\pi(\sigma S_{\alpha'})} \in \tilde{S}. \tag{38}$$

Note that we obtain a different formula for the rank of $E^\alpha_{S_nwS_\alpha} \cap Y_k$ over each point, in particular

$$\mathrm{rank}(E^\alpha_{S_nwS_\alpha} \cap Y_k) = \mathrm{inv}_k(w_-S_\alpha) = \mathrm{inv}_k(w_+S_\alpha) - \mathrm{inv}_\pi(\sigma_0 S_{\alpha'}) \tag{39}$$

where $w_\pm$ are the maximal and minimal elements of S_nwS_α respectively, and $\sigma_0 = (n, ..., 1)$ is the maximal element of S_n.

We now make the following observation: if $U = X - Z$ and all three spaces are paved by affines, then we have that the long exact sequence

$$\cdots \to H^T_*(Z) \to H^T_*(X) \to H^T_*(U) \to H^T_{*-1}(Z) \to \cdots \tag{40}$$

is actually short exact, as the Borel-Moore homology is concentrated in even degree, which in particular implies additivity of characters.

Applying this in our situation, the projection map induces an isomorphism

$$H^{\tilde{T}}_*(E^\alpha_{S_nwS_\alpha} \cap Y^\alpha_k) \cong H^{\tilde{T}}_*(\mathcal{H}^{\alpha'}_\pi) \cong \tilde{S} \otimes_S H^T_*(\mathcal{H}^{\alpha'}_\pi),$$

where the second map exists since loop rotation only acts on the fibers of the bundle. We then have a short exact sequence

$$0 \to H^{\tilde{T}}_*(V^\alpha_Q \cap Y^\alpha_k) \to H^{\tilde{T}}_*(V^\alpha_P \cap Y^\alpha_k) \to H^{\tilde{T}}_*(\mathcal{H}^{\alpha'}_\pi) \to 0 \tag{41}$$

since all terms are nonzero in only even degree.

The following proposition summarizes these statements, and follows by interpreting the results of [GKM03] in type A.

Proposition 4.9. *Let $E^\alpha_{S_nwS_\alpha} = V^\alpha_P - V^\alpha_Q$, $P = Q \cup S_nwS_\alpha \subset W/S_\alpha$, $\alpha' = \alpha(S_nwS_\alpha)$, $w_\pm \in S_nwS_\alpha$ and $\pi = \pi_k(S_nwS_\alpha)$ be as above. Then $Y^\alpha_k \cap E^\alpha_{S_nwS_\alpha}$ is an affine bundle of rank $\mathrm{inv}_k(w_-)$ over the Hessenberg variety $\mathcal{H}^{\alpha'}_\pi$. Moreover, the Schubert cell $\Omega_{\sigma w_- S_\alpha} \cap Y^\alpha_k \cap E^\alpha_{S_nwS_\alpha}$ is identified with the restriction of this bundle to $\Omega_{\sigma S_{\alpha'}} \cap \mathcal{H}_\pi$, and the surjection in* (41) *is given in the fixed point basis by*

$$h^\alpha_{S_nwS_\alpha} : p_{\sigma w_- S_\alpha} \mapsto \frac{\mathrm{LT}_k(\sigma w_- S_\alpha)}{\mathrm{LT}_\pi(\sigma S_{\alpha'})} p_{\sigma S_{\alpha'}} \tag{42}$$

with all other fixed points p_{vS_α} for $vS_\alpha \notin Q$ mapping to zero, which in particular satisfies $h^\alpha_{S_nwS_\alpha}(A^\alpha_{k,\sigma w_- S_\alpha}) = A^{\alpha'}_{\pi,\sigma S_{\alpha'}}$.

Example 4.10. If $w = (1, 6, 2)$, then

$$S_3 w = \{(1, 5, 3), (1, 6, 2), (2, 4, 3), (2, 6, 1), (3, 4, 2), (3, 5, 1)\}$$

and $w_-, w_+ = (2,4,3),(2,6,1)$. We find that $\pi = \pi_1(w_-) = 110100$, $D(\pi) = \{(1,2),(2,3)\}$, and also that $\mathrm{LT}_3(w_-) = x_1 - x_3 + \epsilon$. Inserting this into (42), and plugging in the formulas from (31) and (33), we verify that

$$h_{(2,4,3)}(A_{1,(1,6,2)}) = A_{110100,(3,1,2)},$$

noticing that the ϵ variables cancel.

5 GKM spaces and the nabla operator

In this section we state and prove Theroem A from the introduction, which connects the results of [CM20] with the nil Hecke algebra and the homology of Y_k.

5.1 Connection with the nil Hecke algebra

Consider the ind-subvariety

$$Z = \{V_0 \subset \cdots \subset V_n \in Y : \mathbb{C}[[z]]e_i \subset V_n \text{ for all } i\}. \tag{43}$$

Then the torus fixed points Z are the subset $W_+ \subset W$, which is in fact a lower set in the Bruhat order, and since it is preserved by I, it is a union of Schubert varieties. We have its equivariant homology over the small torus

$$H_*^T(Z) = H_*^T(Y) \cap F \cdot W_+$$

where $F = \mathbb{C}(\mathbf{x})$ be the field of fractions of S, and the intersection is taken in $F \cdot W$. Then from Proposition 4.1 of Section 4.3, we have the explicit description

$$H_*^T(Z) = \bigoplus_{w \in W_+} S \cdot A_w \subset \bigoplus_{d \geqslant 0} \mathbb{A}_{af} \rho^d$$

where $\mathbb{A}_{af}$ is the nil Hecke algebra over S.

Let $M_k = H_*^T(Z_k)$ as an S-submodule of $H_*^T(Z)$. We have that M_k is bigraded by the degree in the x-variables, and also by the positive grading $|w|$ for $w \in W_+$. It is also preserved by the left and right actions of S_n, which preserve the condition in (43). We may therefore consider the Frobenius character

$$\mathcal{F}_{Y,X} M_k \in \Lambda_{q,t}(X,Y).$$

The left and right actions correspond to to the variables Y, X respectively. In other words the symmetric function in the Y-variables keeps track of the dot action, while X corresponds to the choice of parahoric subgroup which must contain the zero label $0 \in J$, in other words come from a composition. We now show that $F_{Y,X} M_k$ is determined by the formula from Theorem 1.1.

Theorem 5.1. *Let $\Delta(\mathbf{x}) = \prod_{i<j}(x_i - x_j)$ be the alternating element.*

(a) We have
$$H_*^T(Y_k) = H_*^T(Y) \cap \Delta(\mathbf{x})^{-k} S \cdot W$$
where the intersection is taken as S-submodules of $F \cdot W$.

(b) The restriction of the surjection $H_*^T(Y_k) \to H_*^T(Y_k^J)$ to the invariants $H_*^T(Y_k)^{W_J}$ is an isomorphism.

(c) The Frobenius character is given by
$$\mathcal{F}_{Y,X} H_*^T(Z_k) = q^{-k\binom{n}{2}} \omega_X \nabla_X^k e_n \left[\frac{XY}{(1-q)(1-t)}\right].$$

Example 5.2. We have
$$[X^{(1,1)}Y^{(1,1)}] q^{-1} \omega_X \nabla_X e_2 \left[\frac{XY}{(1-q)(1-t)}\right] = \frac{1+q^{-1}}{(1-q)^2} + t\frac{1+3q^{-1}}{(1-q)^2} + t^2 \frac{1+5q^{-1}}{(1-q)^2} + \cdots$$

Each coefficient of t^d consists of a shifted copy of part the space shown with paving in Example 4.8, from taking only those fixed points in W_+ which have total degree $|w| = d$.

We need a lemma.

Lemma 5.3. *Let $T = (z_1, ..., z_n)$ be the small torus, with no loop rotation. Then for any element*
$$\sum_w c_w(\mathbf{x}) p_w \in H_*^T(Y_k) \subset F \cdot W, \tag{44}$$
we have that $\Delta(\mathbf{x})^k c_w(\mathbf{x}) \in S$.

Proof. It suffices to prove that for any $v \leqslant_{bru} w$, and any $1 \leqslant i < j \leqslant n$, there is a class $\chi_v \in H^*_T(Y_k \cap V_w)$ which is supported at v, and which only vanishes to degree k at $x_i = x_j$. It thus follows that $f(\mathbf{x})(x_i - x_j)^k c_v = \langle A, \chi_v \rangle \in S$ where A is the element from (44), and $f(\mathbf{x})$ is not divisible by $x_i - x_j$, so that the order with which $x_i - x_j$ enters the denominator of c_v is at most k.

First, consider the case of $n = 2$, and call that Weyl group W'. Then for any w, we have that
$$\chi_v(w) = \begin{cases} (x_1 - x_2)^k & w = v \\ 0 & \text{otherwise} \end{cases}$$

is an element of $H_T^*(Y_k \cap V_w)$. This is seen by noticing that the leading terms in ξ_w divide into $(x_1-x_2)^k$, and that degree coincides with degree of vanishing of x_1-x_2, so that χ_v is related to ξ_w by an upper-triangular change of basis matrix with coefficients in S, and ones on the diagonal.

Now suppose $n > 2$. For any $1 \leqslant i < j \leqslant n$, we have the subgroup $W_{i,j} \subset W$ of all w for which $w_k = k$ provided $k \notin \{i, j\}$, which is generated by $t_{i,j}, t_{j-n,i+n}$, and is isomorphic to the affine Weyl group for $n = 2$. Now, for any v, consider the isomorphism of W-sets which sends the minimal element of $W_{i,j}v$ in the Bruhat order to the identity in W', and let $w' \in W'$ be the image of w for any $w \in W_{i,j}v$. Then we have an element $\tilde{\chi}_v(w) \in \tilde{S}$

$$\tilde{\chi}_v(w) = \begin{cases} \tilde{\chi}'_{v'}(w')\big|_{x_1=x_i, x_2=x_j} & w \in W_{i,j}v \\ 0 & \text{otherwise} \end{cases}$$

where $\tilde{\chi}'_v \in H^*_{\tilde{T}}(Y'_k)$ is any element that lifts $\chi_{v'}$ from the last paragraph to the big torus $\tilde{T}'$. Then using the GKM relations, we see that

$$\left(\prod_{u \in (N^-(v) \cup N^+(v)) - W_{i,j}} \tau(u, v) \right) \tilde{\chi}_v \in H^*_{\tilde{T}}(Y),$$

and maps to the desired element in $H_T^*(Y_k)$ under the specialization $\tilde{S} \to S$.

□

We can now prove the theorem.

Proof. Denote by M the module on the right side in (a). Then the first inclusion $H_*^T(Y_k) \subset M$ follows from Lemma 5.3.

To check the reverse inclusion, we compare the leading terms. Let $P \subset W$ be any lower set, let $w \in P$ be any maximal element, and suppose $Q = P - \{w\}$. Then we have a map $M \cap H_*^T(V_P)/M \cap H_*^T(V_Q) \to S$ which extracts the coefficient of p_w and multiplies by $\Delta(\mathbf{x})^k$, and similarly for $H_*^T(Y_k)$, whose images are given by ideals $I, I' \subset S$ respectively. Then I' is principally generated by $f_k(w) = \Delta(\mathbf{x})^k \operatorname{LT}_k(w)^{-1} \in S$. On the other hand, using the explicit form of (28), we have

$$f_k(w) = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)^{\max\left(k - \# N^+_{i,j}(w), 0\right)}, \tag{45}$$

where $N^+_{i,j}(w)$ is the set of elements $t_{a,b}w \in N^+(w)$ for the affine flag variety, i.e. inversions of w^{-1}, for which a, b are congruent to i, j modulo n. But by the definition of M, each element of I must be divisible by this element, so we have $I = I'$. By induction on the lower set P, we have that $H_*^T(Y_k) = M$. Notice also that $\deg(f_k(w)) = \mathrm{dinv}_k(w)$, using (17).

We have already established (b) in Proposition 4.3.

To prove (c), we have

$$
\begin{aligned}
&[X^\alpha]\mathcal{F}_{Y,X} H_*^T(Z_k) = \mathcal{F}_Y H_*^T(Z_k^\alpha) \\
&= \sum_{S_n w S_\alpha \in S_n\backslash W_+/S_\alpha} t^{|w|} q^{-\operatorname{rank}\left(E^\alpha_{S_n w S_\alpha} \cap Z_k^\alpha\right)} \mathcal{F}_Y H_*^T(\mathcal{H}_\pi^{\alpha'}) \\
&= \sum_{S_n w S_\alpha \in S_n\backslash W_+/S_\alpha} t^{|w|} q^{\#D(\pi) - n'(\alpha') - \mathrm{inv}_k(w_+ S_\alpha)} \mathcal{F}_Y H_*^T(\mathcal{H}_\pi^{\alpha'}) \\
&= \sum_{S_n w S_\alpha \in S_n\backslash W_+/S_\alpha} \frac{t^{|w|} q^{-\mathrm{inv}_k(w_+ S_\alpha)}}{(1-q)^n \mathrm{aut}_q(\alpha')} \xi_\pi[Y;q] \\
&= [X^\alpha] \sum_{[\mathbf{m},\mathbf{a}]} \frac{t^{|\mathbf{m}|} q^{\mathrm{dinv}_k(\mathbf{m},-\mathbf{a}) - k\binom{n}{2}}}{(1-q)^n \mathrm{aut}_q(\mathbf{m},\mathbf{a})} X_{\mathbf{a}} \xi_\pi[Y;q] \\
&= [X^\alpha] \omega_X q^{-k\binom{n}{2}} \nabla_X^k e_n\left[\frac{XY}{(1-q)(1-t)}\right].
\end{aligned}
\tag{46}
$$

In the first line we used item (b) to compute the character in terms of the parabolic. To get to the second line, we used the additivity of Frobenius characters from (40). In the third line we took the second formula for the rank of $E^\alpha_{S_n w S_\alpha} \cap Y_k$ from (39), where $w_+ \in S_n w S_\alpha$ is a maximal element, remembering that $\mathrm{inv}_k(w_+ S_\alpha) = \mathrm{inv}_k(w_-)$ when w_- is a minimal element in $w_+ S_\alpha$. To get the next line, we apply (37), which cancels some of the exponents of q in the numerator. In the second to last line, we apply the rules at the end of Section 3.5, and in the final line we apply Corollary 3.6.

□

6 The lattice action and nabla positivity

We now analyze the action of the lattice $\mathbb{Z}^n$ on $H_*^T(Z_k)$, and present a new conjecture which would categorify (and therefore imply) nabla positivity, Conjecture 3.1.

6.1 The lattice action

We have a lattice $\mathbb{Z}^n \subset W$ generated by the elements $y_i : i \mapsto i + n$, and leaves j fixed when j is not congruent to i modulo n, so that $W = \mathbb{Z}^n \rtimes S_n = S_n \ltimes \mathbb{Z}^n$. The full Weyl group W acts on the left by the dot action on homology, though not on the space level, since the S_n-component permutes the diagonal elements of γ_k. However, the lattice acts on the space level since the translation elements preserve the diagonal matrix γ_k, and in fact the quotient $\mathbb{Z}^n \backslash Y_k$ is a central object from [GKM04], in connection with orbital integrals in general root systems. They studied the homology group $H_*(\mathbb{Z}^n \backslash Y_k)$ from induced action of $\mathbb{C}[y_1^{\pm 1}, ..., y_n^{\pm 1}] = \mathbb{C}[\mathbb{Z}^n]$ on $H_*(Y_k, \mathcal{L})$ with local coefficients.

The positive affine Springer fiber Z_k is only invariant under the semigroup $\mathbb{Z}^n_{\geqslant 0} \subset \mathbb{Z}^n$ generated by y_i. This corresponds to a restricted action of the polynomial algebra $\mathbb{C}[y_1, ..., y_n] \subset \mathbb{C}[y_1^{\pm 1}, ..., y_n^{\pm 1}]$ on $H_*^T(Z_k)$, in which the degree in y_i is the t grading in nabla formulas. We will show directly that this action is free, which has an analogue on the polygraph or "coherent" side mentioned in the introduction.

The space $F \cdot W$ is naturally isomorphic to the algebra $F[y_1^{\pm 1}, ..., y_n^{\pm 1}] \rtimes S_n$. We have

Proposition 6.1. *We have*

1. *The image of $H_*^T(Y)$ in $F[y_1^{\pm 1}, ..., y_n^{\pm 1}] \rtimes S_n$ is the algebra generated by the finite difference operators $A_i = \frac{1-s_i}{x_i - x_{i+1}}$ $(i = 1, \ldots, n-1)$ and the elements $x_i, y_i^{\pm 1}$ $(i = 1, \ldots, n)$.*

2. *The image of $H_*^T(Z)$ in $F[y_1, ..., y_n] \rtimes S_n$ is the algebra generated by A_i $(i = 1, \ldots, n-1)$ and the elements x_i, y_i $(i = 1, \ldots, n)$.*

Proof. The operator A_n can be written in terms of ρ as

$$A_n = \rho A_{n-1} \rho^{-1},$$

whereas ρ can be obtained from

$$\rho = y_1 s_1 \cdots s_{n-1}, \qquad s_i = 1 - (x_i - x_{i+1}) A_i,$$

Proving the first part. To show the second, consider A_w for any $w \in W_+$. If $w = w' s_i$ with $w' < w$ and $1 \leqslant i \leqslant n-1$ we can write $A_w = A_{w'} A_i$, otherwise we would have $0 < w_1 < \ldots < w_n$. If w is not the identity permutation, then $w_n > n$ and we have $w' = w\rho^{-1} \in W_+$, so we can write $A_w = A_{w'} \rho$. □

6.2 The symmetry

Recall that the subspace $H_*^T(Z_k) \subset H_*^T(Z)$ is described as $H_*^T(Z_k) = M_k$, where $M_k = H_*^T(Z) \cap \Delta(\mathbf{x})^{-k} S[y_1, \ldots, y_n] \rtimes S_n$. The Frobenius character of $H_*^T(Y_k)$ as given by Theorem 5.1 when multiplied by $q^{k\binom{n}{2}}$ is a symmetric function in q and t. Thus one may guess that there is some sort of symmetry that interchanges the x and the y variables.

Theorem 6.2. *There is an action of $GL_2(\mathbb{C})$ on M_k such that the diagonal torus action corresponds to the bigrading and the involution $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = y_i.$$

Proof. Consider the differential operators

$$E = \sum_{i=1}^n x_i \frac{\partial}{\partial y_i}, \quad F = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}, \quad H = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} - \sum_{i=1}^n y_i \frac{\partial}{\partial y_i}.$$

Let these operators act on $F[y_1, \ldots, y_n]$ and therefore on $F[y_1, \ldots, y_n] \rtimes S_n$ coefficient-wise. It is straightforward to check that the triple (E, F, H) is a representation of the Lie algebra sl_2, and that these operators preserve $S[y_1, \ldots, y_n] \rtimes S_n$. Let us verify that they also preserve $\Delta(\mathbf{x})^k H_*^T(Z)$. We easily check that

$$[E, A_i] = 0, \qquad [E, x_i] = 0, \qquad [E, y_i] = x_i,$$

$$[H, A_i] = -A_i, \qquad [H, x_i] = x_i, \qquad [H, y_i] = -y_i,$$

$$[F, x_i] = y_i, \qquad [F, y_i] = 0,$$

and with a little bit more work that

$$[F, A_i] = -\frac{y_i - y_{i+1}}{x_i - x_{i+1}} A_i, \qquad \frac{y_i - y_{i+1}}{x_i - x_{i+1}} = A_i y_i - y_{i+1} A_i.$$

Moreover, we have

$$E\Delta(\mathbf{x})^k = \Delta(\mathbf{x})^k E, \qquad H\Delta(\mathbf{x})^k = \Delta(\mathbf{x})^k (H + k\tbinom{n}{2}),$$

$$F\Delta(\mathbf{x})^k = \Delta(\mathbf{x})^k \left(F + k \sum_{i<j} \frac{y_i - y_j}{x_i - x_j} \right).$$

Thus we see that the operators E, F, H preserve both $\Delta(\mathbf{x})^k H_*^T(Z)$ and $S[y_1, \ldots, y_n] \rtimes S_n$, and hence also the intersection $\Delta(\mathbf{x})^k M_k$.

Since the bigrading on $\Delta(\mathbf{x})^k M_k$ has only non-negative degrees, the operators E, F are locally nilpotent, which implies that the action of sl_2 extends to an action of $SL_2(\mathbb{C})$. The center of $GL_2(\mathbb{C})$ acts by the total degree, giving an action of $GL_2(\mathbb{C})$.

Now let us verify the involution property. We begin by writing

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \exp(E) \exp(-F) \exp(E).$$

Then the statement follows from the following computations:

$$\exp(E)x_i = x_i \exp(E), \qquad \exp(-F)x_i = (x_i - y_i)\exp(-F),$$

$$\exp(E)(x_i - y_i) = -y_i \exp(E), \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y_i = -y_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

□

Since M_k is free over S we obtain

Corollary 6.3. *We have that $H_*^T(Z_k)$ is free as a module over $\mathbb{C}[\mathbf{y}]$.*

Corollary 6.4. *We have that*

$$\mathcal{F}_{Y,X}(N_k) = q^{-k\binom{n}{2}} \omega_X \nabla^k e_n \left[\frac{XY}{1-q}\right]. \tag{47}$$

where $N_k = H_*^T(Z_k)/(y_1, ..., y_n)H_*^T(Z_k)$.

Proof. This follows by applying the plethystic substitution $Y \mapsto Y(1-t)$ to both sides of Theorem 5.1, item (c). □

6.3 Koszul submodules

We begin by describing the 2^n submodules

$$\mathbf{y}^{\mathbf{s}} H_*^T(Z_k) = (y_1^{s_1} \cdots y_n^{s_n}) H_*^T(Z_k)$$

for $s_i \in \{0, 1\}$, which constitute the terms in the Koszul resolution, as the homologies of certain closed ind-subvarieties $Z_k^{\mathbf{s}} \subset Z_k$ as follows.

Definition 6.5. For each $\mathbf{s} = (s_1, ..., s_n) \in \{0,1\}^n$, define

$$Z^{\mathbf{s}} = \left\{V_0 \subset \cdots \subset V_n \in Y : z^{-s_i}\mathbb{C}[[z]]e_i \in V_n\right\}$$

considered as an ind-variety

$$Z^{\mathbf{s}} = \lim_{\rightarrow} Z^{\mathbf{s}} \cap V_w$$

over all Schubert varieties $V_w \subset Z$. Let $Z_k^{\mathbf{s}} = Z^{\mathbf{s}} \cap Z_k$ be the intersection with the affine Springer fiber, also as an ind-variety, and let $Z_k^i = Z_k^{\mathbf{s}}$ where $\mathbf{s}$ is defined by $s_i = 1$, and all other entries are zero. In particular, we have $Z_k^{0^n} = Z_k$, and the fixed point set is given by $(Z_k^{\mathbf{s}})^T = W_{\mathbf{s}}$ where

$$W_{\mathbf{s}} = \mathbf{y}^{\mathbf{s}} W_+ = \left\{w \in W_+ : w_i^{-1}/n \leqslant 1 - s_i \text{ for } 1 \leqslant i \leqslant n\right\}. \tag{48}$$

We have the following proposition.

Proposition 6.6. *Let $P \subset W_+$ be a lower set which is invariant under the left S_n action, and let $V_P \subset Z$ be the corresponding union of Schubert varieties. Then we have*

1. *Each $Z_k^{\mathbf{s}} \cap V_P$ is a GKM subvariety of Z_k with respect to the big torus $\tilde{T}$ and we have $H_*^T(Z_k^{\mathbf{s}} \cap V_P) = H_*^T(Z_k^{\mathbf{s}}) \cap H_*^T(V_P)$.*

2. *If $Q \subset P$ and $\mathbf{t} \geqslant \mathbf{s}$ componentwise, then $H_*^T(Z_k^{\mathbf{t}} \cap V_Q) \hookrightarrow H_*^T(Z_k^{\mathbf{s}} \cap V_P)$ is injective, and the image splits as a direct summand of S-modules.*

3. *The image of $H_*^T(Z_k^{\mathbf{s}}) \hookrightarrow H_*^T(Z_k)$ is given by $\mathbf{y}^{\mathbf{s}} H_*^T(Z_k)$.*

Proof. To prove the GKM statement, it suffices to assume $\mathbf{s} = (1^l 0^{n-l})$ with all ones on the left, using the left S_n-symmetry. Then $Z_k^{\mathbf{s}}$ is a union of Schubert varieties, as it is isomorphic to $\rho^k Z_k$ via the rotation operator ρ on Y_k, which preserves the Bruhat order. In other words, each of the $Z_k^{\mathbf{s}}$ is a union of Schubert varieties rotated by some element of S_n. Hence $Z_k^{\mathbf{s}} \cap V_P$ is also a union of Schubert varieties and therefore is GKM. The homology of $Z_k^{\mathbf{s}} \cap V_P$ is spanned by the classes of cells which belong both to $Z_k^{\mathbf{s}}$ and V_P, and since the classes of cells form a free basis of $H_*^T(Z_k)$ we see that $H_*^T(Z_k^{\mathbf{s}} \cap V_P) = H_*^T(Z_k^{\mathbf{s}}) \cap H_*^T(V_P)$.

For the second statement we may assume that $Q \subset P$, $t_i \geqslant s_i$, and $\mathbf{s}, \mathbf{t}$ are both sorted so that the ones are all on the left, using the S_n action simultaneously. In this case both $Z_k^{\mathbf{s}} \cap V_P, Z_k^{\mathbf{t}} \cap V_Q$ are unions of Schubert varieties, which determines the splitting.

For the final statement, note that both homologies embed into $H_*^T(Y_k)$ on which $\mathbf{y}^{\mathbf{s}}$ acts as an automorphism. The automorphism comes from the geometric automorphism which sends Z_k to $Z_k^{\mathbf{s}}$. Thus $\mathbf{y}^{\mathbf{s}}$ bijectively maps $H_*^T(Z_k)$ to $H_*^T(Z_k^{\mathbf{s}})$.

□

Let $\mathcal{Z}_k = \{Z_k^{\mathbf{s}} \cap V_P \subset Z_k\}$ be the collection of all subspaces appearing in the above proposition, for *finite* lower sets $P \subset W_+$, which is itself a poset by inclusion, and is closed under taking intersections. By the above proposition, the homology $H_*^T(Z_k^{\mathbf{s}} \cap V_P) \subset H_*^T(Z_k)$ is a free S-module.

Proposition 6.7. *The submodules of the form $H_*^T(Z_k^{\mathbf{s}} \cap V_P) \subset H_*^T(Z_k)$ generate a distributive lattice. For any union of elements of $\mathcal{Z}_k$ we have*

$$H_*^T((Z_k^{\mathbf{s}_1} \cap V_{P_1}) \cup \cdots \cup (Z_k^{\mathbf{s}_m} \cap V_{P_m})) =$$

$$(H_*^T(Z_k^{\mathbf{s}_1}) \cap H_*^T(V_{P_1})) + \cdots + (H_*^T(Z_k^{\mathbf{s}_m}) \cap H_*^T(V_{P_m})).$$

Proof. Note that the two statements are essentially equivalent (see Proposition A.8). In fact we will prove the statement for more general subsets of the form $Z_k^{\mathbf{s}} \cap V_P$, where $P \subset W_+$ is a lower set in the Bruhat order, which is not necessarily S_n-invariant.

The set Z_k is paved in two ways: in one way by Bruhat cells, in another way by the 2^n subsets of the form

$$Z_k^{\mathbf{s}\,o} := Z_k^{\mathbf{s}} \backslash \bigcup_{\substack{\mathbf{t} \geqslant \mathbf{s} \\ \mathbf{t} \neq \mathbf{s}}} Z_k^{\mathbf{t}}.$$

Consider the paving obtained by intersecting these two pavings. A cell in the new paving is the intersection of the Schubert cell $IwI/I \cap Y_k$ corresponding to some $w \in W_+$ with the set $Z_k^{\mathbf{s}\,o}$ for some $\mathbf{s} \in \{0,1\}^n$. To complete the proof it sufficient to show that the equivariant homology of these intersections is concentrated in even degrees (see Proposition A.9).

Explicitly, the Schubert cell IwI/I is identified with the unipotent Lie group generated by elements of the form $1 + E_{i,j}$ for $1 \leqslant i \leqslant n$, $i < j$, $w_i^{-1} > w_j^{-1}$, where $E_{i,j}$ is the matrix with $t^{\lceil \frac{j}{n} \rceil - 1}$ at position $i, j \mod n$ and other entries zero. Let us parametrize the points on IwI/I as follows:

$$v(\mathbf{u}) := \left(1 + \sum_{i,j} u_{i,j} E_{i,j}\right)^{-1} wI,$$

where $u_{i,j} \in \mathbb{C}$ are coordinates. The condition that the element is preserved by $1 + \gamma_k$ results in a system of equations of the form

$$u_{i,j} = \text{polynomial in } u_{i',j'} \text{ with } j' - i' < j - i$$

for each pair (i, j) satisfying $j > i + kn$. Thus to coordinatize the intersection $IwI/I \cap Y_k$ we can simply eliminate the variables $u_{i,j}$ with $j > i + kn$ using the obtained equations.

Next we pick r, $1 \leqslant r \leqslant n$ and analyze the condition that $v(\mathbf{u}) \in Z_k^r$. By definition, this means

$$t^{-1}e_r \in \left(1 + \sum_{i,j} u_{i,j}E_{i,j}\right)^{-1} w\mathcal{O}^n,$$

equivalently

$$\left(1 + \sum_{i,j} u_{i,j}E_{i,j}\right) e_r \in t\sum_{i=1}^{n} \mathcal{O}e_{w_i}.$$

The left hand side can be written as follows:

$$e_r + \sum_{i,l} u_{i,r+ln}t^l e_i.$$

Since $\mathcal{O} \subset \mathcal{O}e_{w_i}$, the terms with $l > 0$ can be ignored. We see that the condition $v(\mathbf{u}) \in Z_k^r$ is equivalent to the following: for each $1 \leqslant i \leqslant n$ satisfying $w_i \leqslant n$ we have $w_i \neq r$ and $u_{w_i,r} = 0$. Recall that $u_{i,j}$ makes sense only when $i < j$ and $w^{-1}(i) > w^{-1}(j)$. We then have:

$v(\mathbf{u}) \in Z_k^r$ if and only if $w_r^{-1} \leqslant 0$ and for each $1 \leqslant i < r$ such that $w_i^{-1} \geqslant 1$ we have $u_{i,r} = 0$. Denote the set of such indices i by A_r.

Notice that the condition only involves variables $u_{i,r}$ with $r \leqslant n$, and so it does not depend on the variables we eliminated when passing from IwI/I to $IwI/I \cap Y_k$.

It is now clear that the intersection of $IwI/I \cap Y_k$ with $Z_k^{\mathbf{s}\,o}$ if not empty is always of the form

$$U = \mathbb{C}^N \times \prod_{r:\, s_r=0, w_r^{-1}\leqslant 0} \mathbb{C}^{d_r}\backslash\{0\} = \mathbb{A}\backslash\bigcup_r H_r$$

for $d_r = |A_r|$. Here $\mathbb{A}$ is an affine space and H_r is a coordinate subspace of codimension d_r. The T-characters that appear in $\mathbb{C}^{d_r}$ correspond to differences of the form $x_i - x_r$ with $i \in A_r$, $i < r$.

We now have an explicit description

$$H_*^T(U) = \mathbb{C}[x_1, \ldots, x_n]/\left(\prod_{i\in A_r}(x_i - x_r) \mid s_r = 0, w_r^{-1} \leqslant 0\right).$$

This can be seen as follows: first identify $H_*^T(\mathbb{A})$ with the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Then the space U is the complement of a subspace arrangement, and the homology of H_r is the ideal generated by the polynomial $f_r = \prod_{i\in A_r}(x_i - x_r)$. The homology of an arbitrary intersection is the ideal generated by the corresponding product of polynomials. The polynomials f_r clearly form a regular sequence: each subsequent polynomial has as a leading term which is a power of a variable that does not appear in previous polynomials. By Proposition A.6 we conclude that the homologies of H_r generate a distributive lattice. By Proposition A.8 we conclude that $H_*^T(\bigcup_r H_r)$ is the ideal generated by all f_r, and from the long exact sequence we see that $H_*^T(U)$ is the quotient.

So we have shown that the space Z_k is paved by cells whose equivariant homology is supported in even degrees. Every set of the form $Z_k^{\mathbf{s}} \cap V_P$ is a union of cells corresponding to a lower set, so Proposition A.9 completes the proof. □

Remark 6.8. Combining Proposition 6.7 with Proposition A.6 we obtain another proof of Corollary 6.3.

We now have a geometric description of N_k:

Definition 6.9. Let $U_k = Z_k - Z_k^1 \cup \cdots \cup Z_k^n$, which is open in Z_k

Corollary 6.10. *We have that*

$$H_*^T(U_k) = H_*^T(Z_k)/\left(H_*^T(Z_k^1) + \cdots + H_*^T(Z_k^n)\right) = N_k \tag{49}$$

In particular, $H_i^T(U_k) = 0$ *for* i *odd.*

Proof. We have the long exact sequence in equivariant Borel-Moore homology:

$$\cdots \to H_i^T(Z_k^1 \cup \cdots \cup Z_k^n) \to H_i^T(Z_k) \to H_i^T(U_k) \to \cdots \tag{50}$$

Now apply Proposition 6.7 to expand the homology of the union as a sum, and also to see that the first map is injective. □

Remark 6.11. It follows by the previous corollaries that

$$H_*^T(U_k) = H_*^T(Z_k) \otimes_{\mathbb{C}[\mathbf{y}]} \mathbb{C}$$

where $\mathbb{C}$ is the $\mathbb{C}[\mathbf{y}]$ module on which each y_i acts by zero.

Remark 6.12. Note that despite the vanishing of odd equivariant homology, U_k is not equivariantly formal, and in fact has odd nonequivariant homology. Indeed, equivariant formality would imply that $N_k \cong H_*^T(U_k)$ is free over $\mathbb{C}[\mathbf{x}]$, which it is not.

Remark 6.13. Since the space U_k is paved by spaces whose equivariant Borel-Moore homology is pure, the Hodge structure on $H_*^T(U_k)$ is also pure. Thus by [FW05] the Eilenberg-Moore spectral sequence $\mathrm{Tor}_i^S(H_{-j}^T(U_k), \mathbb{C}) \Rightarrow H_{-j-i}^*(U_k)$ degenerates.

Remark 6.14. Since $H_*^T(Z_k^i)$ generate a distributive lattice, the following complex is a free resolution of $H_*^T(U_k)$ over S:

$$\cdots \to \bigoplus_{i<j} H_*^T(Z_k^i \cap Z_k^j) \to \bigoplus_i H_*^T(Z_k^i) \to H_*^T(Z_k).$$

Therefore Tor is computed by the complex

$$\cdots \to \bigoplus_{i<j} H_*(Z_k^i \cap Z_k^j) \to \bigoplus_i H_*(Z_k^i) \to H_*(Z_k), \tag{51}$$

since the intersections of Z_k^i are equivariantly formal (Proposition 6.6 (i)). The complex (51) can now be viewed as the E_1 page of the spectral sequence converging to $H_*(U_k)$. Since the Borel-Moore homologies of the intersections of Z_k^i are pure, this spectral sequence degenerates at the E_2 page.

Using either Remark 6.13 or Remark 6.14 we conclude with the following description of the weight filtration on $H_*(U_k)$:

Corollary 6.15. *We have*

$$\mathrm{Gr}^W_{i-*} H_*(U_k) = \mathrm{Tor}_i^S(H_*^T(U_k), \mathbb{C}).$$

where Gr_i^W *denotes the associated graded for the weight filtration, i.e.* $\mathrm{Tor}_0^S(H_*^T(U_k), \mathbb{C})$ *equals to the pure part of the homology,* $\mathrm{Tor}_1^S(H_*^T(U_k), \mathbb{C})$ *equals to the part one degree off from pure and so on.*

6.4 Open Hessenberg varieties

We now study the intersection of the Hessenberg paving of Section 4.6 with U_k. We have a subvariety $Z^{\mathbf{s}}_{\pi,l} = Z^{\mathbf{s}}_{n,l} \cap \mathcal{H}_\pi$, where

$$Z^{\mathbf{s}}_{n,l} = \{V_1 \subset \cdots \subset V_n : s_i = 1 \Rightarrow e_i \in V_{n-l}\} \tag{52}$$

and the intersection is taken in the usual flag variety Fl_n, with a similar definition for $Z^i_{\pi,l}$ where $\mathbf{s}$ has a one in position i as above. We consider the complementary variety to the $Z^i_{\pi,l}$:

Definition 6.16. Let π be a Dyck path, and $0 \leqslant l \leqslant n$. Let $U_{\pi,l} = U_{n,l} \cap \mathcal{H}_\pi$, where

$$U_{n,l} \cong \{V_1 \subset \cdots \subset V_n : e_i \notin V_{n-l} \text{ for any } i\} \tag{53}$$

which is the complement of the $Z^i_{\pi,l}$ in $\mathcal{H}_\pi$.

Remark 6.17. We may also replace the definition of $U_{n,l}$ by taking orthogonal complements in Fl_n:

$$\{V_1 \subset \cdots \subset V_n : V_l \not\subset H_i \text{ for any } i\},$$

where H_i is the *axis hyperplane* perpendicular to e_i, and the isomorphism is induced by taking orthogonal complements. This may be more natural since it generalizes the construction of hyperplane complements in $\mathbb{CP}^n$. In this case the cohomology ring $H_T^*(U_l)$ generalizes the (equivariant) Orlik-Solomon algebra.

Let $P = Q \cup S_n w$ as in Section 4.6, but now supposing that $P, Q \subset W_+$, and let $\pi = \pi_k(S_n w)$.

Proposition 6.18. *We have that $U_k \cap (V_P - V_Q)$ is the restriction of $E_w \cap Y_k$ as an affine bundle over $\mathcal{H}_\pi$ to $U_{\pi,l}$, where l is the number of indices i with $w_i \in \{1, ..., n\}$.*

Proof. We have that $U_{\pi,l} = \mathcal{H}_\pi - Z^1_{\pi,l} \cup \cdots \cup Z^n_{\pi,l}$, so it suffices to check that $Z^{\mathbf{s}}_k \cap V_P - Z^{\mathbf{s}}_k \cap V_Q$ is the restriction of $E_w \cap Y_k$ to $Z^{\mathbf{s}}_\pi$ for any $\mathbf{s}$. It suffices to consider $\mathbf{s} = (1^m 0^{n-m})$, in which $Z^{\mathbf{s}}_{\pi,l}$ is a union of Hessenberg-Schubert cells. This now follows from the description of the Schubert cells in Proposition 4.9. □

One can ask if all possible pairs (π, l) can appear in the above proposition. It is not hard to see that the number of indices with $w_i \in \{1, ..., n\}$, which is an invariant of the coset $S_n w$, is at most the number of trailing East steps in $\pi_k(S_n w)$. In terms of labels $[\mathbf{m}, \mathbf{a}]$, this number is the same as the number of zeroes in $\mathbf{m}$, and will be denoted $z(\mathbf{m}) = z(S_n w)$ for $w = \mathrm{aff}(\mathbf{m}, \mathbf{a})$. Thus, the data of possible (π, l) is the same as that of a partial Dyck path, discussed in Section 3.5. We will write $\pi' = (\pi, l)$ for a partial Dyck path and write $U_{\pi'} = U_{\pi,l}$. We will also use the notation $\pi'_k(S_n w) = (\pi_k(S_n w), z(S_n w))$, and similarly for $\pi'_k(\mathbf{m}, \mathbf{a})$.

We have an explicit description of the equivariant cohomology $H_T^*(U_{\pi,l})$, which agrees with Borel-Moore homology since $U_{\pi,l}$ is smooth. Let $M_\pi = H_T^*(\mathcal{H}_\pi)$, which is determined by the GKM relations. Let

$$A^{\mathbf{s}}_{n,l} = \left\{\sigma \in S_n : s_i = 1 \Rightarrow \sigma_i^{-1} \leqslant n - l\right\}$$

be the subset of fixed points of $Z^{\mathbf{s}}_{\pi,l}$, and similarly for $A^i_{n,l}$ and $Z^i_{n,l}$. We define

$$N_{\pi,l} = M_\pi / \left(F_{A^1_{n,l}} M_\pi + \cdots + F_{A^n_{n,l}} M_\pi\right), \tag{54}$$

where F_A denotes the space of elements supported at the fixed points in A.

Proposition 6.19. *We have that $H_T^*(U_{\pi'}) \cong N_{\pi'}$.*

Proof. Since the varieties are S_n rotations of Schubert varieties, we have that $F_{A_{i,l}} M_\pi$ is the image of $H_*^T(Z_{\pi,l}^i)$ under Poincaré duality. Then we proceed in the same way as Corollary 6.10, noting that $Z_{\pi,l}^i$ also satisfy the lattice property of Proposition 6.7. □

Remarks 6.13 and 6.14 can be repeated for the spaces $U_{\pi'}$ and we obtain the following analogue of Corollary 6.15, where we can replace Borel-Moore homology with cohomology since $U_{\pi'}$ is smooth:

Corollary 6.20. *We have*

$$\mathrm{Gr}^W_{i+*} H^*(U_{\pi'}) = \mathrm{Tor}_i^S(H_T^*(U_{\pi'}), \mathbb{C}),$$

i.e. Tor_0 *equals to the pure part of the cohomology,* Tor_1 *equals to the part one degree off from pure and so on.*

6.5 Conjectures about Tor groups

We now present some conjectures that would categorify the nabla positivity conjecture.

Definition 6.21. Let N be a module over $S = \mathbb{C}[\mathbf{x}]$ with a left action of S_n, which intertwines the action on S. We will say that N "satisfies the Tor property" if the multiplicity of the irreducible representation χ_λ in

$$S_n \circlearrowleft \mathrm{Tor}_i^S(N, \mathbb{C})$$

is zero unless $i = \iota(\lambda)$, the number of boxes in λ below the main diagonal (see Conjecture 3.1).

We now connect these modules to the nabla positivity conjecture, which is Conjecture 3.1.

Conjecture 6.22. Both $H_*^T(U_k)$ and $H_*^T(U_{\pi'})$ satisfy the Tor property as modules over S.

Note that we do not conjecture this property for $U_{\pi,l}$ when l is greater than the number of trailing East steps, in which case the conjecture does not appear to hold.

Theorem 6.23. *The Tor property for* $H_*^T(U_k)$ *implies nabla positivity, Conjecture 3.1. The Tor property for* $H_*^T(U_{\pi'})$ *implies the Tor property for* $H_*^T(U_k)$*.*

Proof. For the first statement, we apply the plethystic substitution $Y \mapsto Y(1-q)$ to both sides of (47) to get

$$\sum_{i\geqslant 0}(-1)^i \mathcal{F}_{Y,X}\operatorname{Tor}_i^S(N_k,\mathbb{C}) = \omega_X \nabla^k e_n[XY] =$$

$$\sum_{\lambda} \omega_X \nabla^k s_{\lambda'}(X) s_\lambda(Y) = \sum_{\lambda,\mu} c_{\lambda',\mu'}(q,t) s_\mu(X) s_\lambda(Y). \tag{55}$$

It follows that $c_{\lambda',\mu}(q,t)$ is signed positive, since only $i = \iota(\lambda)$ contributes in (55).

For the second statement, let $P = Q \cup S_n w \subset W_+$ be lower sets as above. Then since (50) is short exact by Corollary 6.10, we have the long exact sequence

$$\cdots \to \operatorname{Tor}_i^S(H_*^T(U_k \cap V_Q),\mathbb{C}) \to \operatorname{Tor}_i^S(H_*^T(U_k \cap V_P),\mathbb{C}) \to \operatorname{Tor}_i^S(H_*^T(U_{\pi,l}),\mathbb{C}) \to \cdots$$

Where $(\pi, l) = \pi'_k(S_n w)$. The vanishing then follows by induction on P.

□

6.6 A formula for the Frobenius character

We now compute the Frobenius character $\mathcal{F}_{-Y} N_{\pi'}$. Given a label $\mathbf{b}$ and a composition $\mathbf{t} \in \{0,1\}^n$, define a new composition by

$$(\mathbf{b}\cdot\mathbf{t})_i = (-1)^{t_i}(|b_i| + t_i N), \tag{56}$$

where $N = \max(|b_1|, ..., |b_n|)$. Here the meaning of the signs refer to the super variables described in Section 3.3.

Definition 6.24. We have a symmetric function

$$\chi_{\pi,l}[Y;q] = \sum_{\mathbf{b},\mathbf{t},i>n-l\Rightarrow t_i=0} (-1)^{|\mathbf{t}|} q^{\operatorname{inv}_\pi(\mathbf{b}\cdot\mathbf{t})} Y_{\mathbf{b}}. \tag{57}$$

Proposition 6.25. *We have*

$$\mathcal{F}_Y H_T^*(U_{\pi,l}) = \frac{1}{(1-q)^n} \chi_{\pi,l}[Y;q].$$

Proof. Proceeding the same way as Corollary 6.3, we see that there is an exact sequence

$$0 \to E_{n-l} \to \cdots \to E_0 \to H_*^T(U_{\pi,l}) \to 0, \quad E_m = \bigoplus_{|\mathbf{s}|=m} H_*^T(Z_{\pi,l}^{\mathbf{s}})$$

of S-modules, where the maps are the same as the differentials in Çech cohomology using the maps induced from the inclusion $H^T_*(Z^{\mathbf{s}}_{\pi,l})$. The maps are equivariant once the S_n-action is twisted by the sign representation on the S_m-factor, and we have

$$E_m \cong \mathrm{Ind}^{S_n}_{S_m\times S_{n-m}} H^T_*(Z^{\mathbf{s}}_{\pi,l}) \otimes (\mathrm{sgn}_m \boxtimes \mathrm{triv}_{n-m})$$

when $\mathbf{s} = (1^m 0^{n-m})$ ones, noticing that in this case $Z^{\mathbf{s}}_{\pi,l}$ is preserved by $S_m \times S_{n-m}$, as is the fixed point set $A^{\mathbf{s}}_{n,l}$.

We must therefore compute the Frobenius character of the resolution. We have

$$\left((\mathrm{sgn}_m \boxtimes \mathrm{triv}_{n-m}) H^T_*(Z^{\mathbf{s}}_{\pi,l})\right)^{S_{\alpha'}\times S_{\alpha''}} = \frac{q^{-\#D(\pi)}}{(1-q)^n} \sum_{\sigma\in S_\alpha\times S_{\alpha'}\backslash A^{\mathbf{s}}_{n,l}} q^{\mathrm{inv}_\pi(\sigma)} \tag{58}$$

where α, α' are compositions of $m, n-m$ respectively, the (anti) invariants are with respect to the dot action, and σ is identified with the minimal representative. This follows since $Z^{\mathbf{s}}_{\pi,l}$ is a union of Hessenberg Schubert varieties and so the character can be computed using the corresponding subset of summands of (34). As a quasi-symmetric function this is

$$\mathcal{F}_{Y_1,Y_0} H^T_*(Z^{\mathbf{s}}_{\pi,l}) = \frac{q^{-\#D(\pi)}}{(1-q)^n} \sum_{\mathbf{b},\mathbf{t},i>n-l\Rightarrow t_i=0} q^{\mathrm{inv}_\pi(\mathbf{b})-\#D(\pi)} Y_{\mathbf{t},\mathbf{b}}$$

where $Y_{\mathbf{t},\mathbf{b}}$ is the product of Y_{t_i,b_i}, which is symmetric in both sets Y_1, Y_0. But in terms of Frobenius characters, induction from $S_m \times S_{n-m}$ combined with the sign twist corresponds to evaluating $Y_1 = -Y, Y_0 = Y$, and the result follows.

□

Corollary 6.26. *We have that*

$$\nabla^k e_n\left[\frac{XY}{1-q}\right] = \sum_{[\mathbf{m},\mathbf{a}]} \frac{t^{|\mathbf{m}|} q^{\mathrm{dinv}_k(\mathbf{m},\mathbf{a})}}{(1-q)^n \mathrm{aut}_q(\mathbf{m},\mathbf{a})} X_{\mathbf{a}} \chi_{\pi'_k(\mathbf{m},\mathbf{a})}[Y;q]. \tag{59}$$

We list some basic properties of $\chi_{\pi'}$ and give an example.

Proposition 6.27. *We have $\chi_{\pi,l} = 0$ for $l = 0$.*

Proof. In this case there is no restriction on the values of t_i, and we find that $\chi_{\pi,0}[Y;q] = \xi_\pi[Y-Y;q] = 0$. Obviously, in this case $U_{\pi,l}$ is empty. □

Proposition 6.28. *We have $\chi_{\pi,l} = 0$ if there exists $1 \leqslant i < n$ such that $1,\dots,i$ do not attack $i+1,\dots,n$. In other words, π has "touch points," where the path contacts the diagonal.*

Proof. Note that $n-l+1$ attacks n, so $i \leqslant n-l$. It follows that $\chi_{\pi,l}$ factors as $\chi_{\pi,l} = \chi_{\pi_1,0}\chi_{\pi_2,l}$, where π_1 the beginning part of π of length i and π_2 are the remaining steps. We have $\chi_{\pi_1,0} = 0$ by Proposition 6.27, hence $\chi_{\pi,l} = 0$. Note that in this case points of $\mathcal{H}_\pi$ satisfy $\gamma V_i = V_i$ and therefore $V_i \subset V_{n-l}$ must contain a basis vector, so $U_{\pi,l}$ is empty. □

It is not hard to prove the following:

Proposition 6.29. *Let* $(\mathbf{m}, \mathbf{a})$ *be sorted, and let* $\mathbf{m}' = \mathbf{m}_{\mathrm{Std}(\mathbf{a})^{-1}}$, $(\pi, l) = \pi'(\mathbf{m}, \mathbf{a})$. *Then the following are equivalent:*

1. *There exists* $\tau \in S_n$ *such that* $\mathbf{m}'$ *is the exponent in the Garsia-Stanton descent polynomial* $g_\tau(\mathbf{y}) = \mathbf{y}^{\mathbf{m}'}$, *whose degree is* $\mathrm{maj}(\tau)$.
2. *We have* $l > 0$ *and* π *has no touch points.*

Combining Propositions 6.27, 6.28, and 6.29, we have:

Corollary 6.30. *The sum in* (59) *has at most* $n!$ *nonzero terms.*

We can now give an example of one of the sums from Corollary 6.26.

Example 6.31. Using Proposition 6.25, we evaluate

$$(1-q)^3 \nabla_X e_3 \left[\frac{XY}{1-q}\right] = M_{(3)} \frac{1}{(1+q)\left(1+q+q^2\right)} \chi_{111} +$$

$$M_{(2,1)} \left(\frac{t}{1+q}\chi_{1110} + \frac{1}{1+q}\chi_{111} + t^2\chi_{11010}\right) +$$

$$M_{(1,2)} \left(t\chi_{1101} + \frac{1}{1+q}\chi_{111} + \frac{t^2}{1+q}\chi_{11100}\right) +$$

$$M_{(1,1,1)} \left(\left(t^2+t^3\right)\chi_{11010} + t\chi_{1110} + t\chi_{1101} + \chi_{111} + t^2\chi_{11100}\right).$$

Here $M_\alpha = M_\alpha(X)$ is the quasi-symmetric monomial in the X variables, and have eliminated all terms with no contribution using Corollary 6.30.

Now writing $\chi'_{\pi'} = (1-q)^{-n}\chi_{\pi'}[Y(1-q); q]$, we get

$$\chi'_{1101} = (1+q)\, s_3 - q^2 s_{2,1}$$

$$\chi'_{11010} = q^2 s_{111} - q s_{2,1} + s_3$$

$$\chi'_{11100} = (1+q)\, s_3 - q\,(1+q)\, s_{2,1} + q^2\,(1+q)\, s_{1,1,1}$$

$$\chi'_{111} = (1+q)\left(1+q+q^2\right)s_3$$
$$\chi'_{1110} = (1+q)^2 s_3 - q^2(1+q)s_{2,1}.$$

Notice that each factor satisfies the signed positivity property, and is divisible by the automorphism factors in the above equation. Moreover, the factors in front of $M_{(1,2)}$ and $M_{(2,1)}$ are equal, so we do in fact end up with a symmetric function.

Example 6.32. Conjecture 6.22 can be used to compute the cohomology groups of the $U_{\pi'}$ and U_k, and we give an example for $U_{\pi'}$: we calculate

$$\chi_{11100110} = -(1+q)^2(q-1)^3 s_5 - q(1+q)^2(q-1)^3 s_{4,1}.$$

Recall that Borel-Moore homology agrees with cohomology for smooth spaces, so we have $H_T^*(U_{\pi'}) = H_*^T(U_{\pi'})$. From this we compute the signed Frobenius character of $H_*(U_{\pi'})$, by applying the substitution

$$F[Y] \mapsto (1-q)^{-n}F[(1-q)Y]\big|_{s_\lambda = x^{-\iota(\lambda)}s_\lambda, q=x^2},$$

where x is the generating variable for degree. We obtain

$$\left(x^2+1\right)^3 s_5 - x^3\left(x^2+1\right)^2 s_{4,1} - x^3\left(x^2+1\right)^2 s_{3,2} +$$
$$x^4\left(x^2+1\right)^2 s_{3,1,1} + x^4\left(x^2+1\right)^2 s_{2,2,1} - \left(x^2+1\right)^2 x^5 s_{2,1,1,1},$$

noticing that the signs respect the parity of the degree in x.

A Distributive lattices

We collect some useful facts and definitions about collections of subspaces of a fixed vector space here.

Definition A.1. Let $\mathcal{L}$ be a collection of subspaces of a fixed vector space $\mathcal{V}$. We say $\mathcal{L}$ is a *lattice* if for any subspaces $A, B \in \mathcal{L}$ we have $A \cap B, A+B \in \mathcal{L}$. If $\mathcal{L}$ is not necessarily a lattice, we can always form a lattice by taking all possible combinations of the operations $\cap$ and $+$ and applying them to all collections of elements of $\mathcal{L}$. A lattice thus obtained is called the *lattice generated by* $\mathcal{L}$. A lattice $\mathcal{L}$ is called *distributive* if for any A, B, C we have

$$(A+B)\cap C = (A\cap C) + (B\cap C).$$

The condition above can be replaced by the condition

$$(A \cap B) + C = (A + C) \cap (B + C).$$

Definition A.2. Given a vector space $\mathcal{V}$ with a choice of a basis $\{e_i\}_{i \in I}$, a subspace $A \subset \mathcal{V}$ is called a *coordinate subspace* if there exists a subset $J \subset I$ such that A is the span of $\{e_i\}_{i \in J}$.

It is clear that all coordinate subspaces of a given vector space with a given basis form a distributive lattice. Conversely, we have

Proposition A.3. *Suppose $\mathcal{L}$ is a finite distributive lattice of subspaces of $\mathcal{V}$. There exists a basis of $\mathcal{V}$ such that all elements of $\mathcal{L}$ are coordinate subspaces.*

Proof. Replacing $\mathcal{V}$ by $\sum_{A \in \mathcal{L}} A$ or by $\mathcal{L}/\bigcap_{A \in \mathcal{L}} A$ if necessary we may assume that $\{0\}, \mathcal{V}$ are both in $\mathcal{L}$. Now we induct on the number of elements of $\mathcal{L}$. The base case $|\mathcal{L}| = 2$ is obvious. For the induction step, let $A \in \mathcal{L}$ be any maximal element of $\mathcal{L}$ not equal to $\mathcal{V}$, and let B be any minimal element of $\mathcal{L}$ not contained in A. Note that we have $A + B = \mathcal{V}$ by the maximality of A. Elements of $\mathcal{L}$ contained in A form a distributive lattice. Applying the induction assumption, choose any basis of A such that all elements of $\mathcal{L}$ contained in A are coordinate subspaces, so in particular $A \cap B$ is a coordinate subspace. Complete the basis of $A \cap B$ to a basis of B, and we obtain a basis of $\mathcal{V}$ such that any element of $\mathcal{L}$ contained in A is a coordinate subspace, and B is a coordinate subspace. Now let $C \in \mathcal{L}$ be any element and let us prove that C is a coordinate subspace. If $C \subset A$ we are done. Otherwise, apply distributivity to write

$$C = C \cap (A + B) = (C \cap A) + (C \cap B).$$

Here $C \cap B \subset B$ is such that $C \cap B \not\subset A$. By the minimality of B we have $C \cap B = B$, so we have $C = (C \cap A) + B$. Since both $C \cap A$ and B are coordinate subspaces, C is as well. □

Remark A.4. The statement is not true in general for infinite lattices. For a counterexample, let $\mathcal{V} = \mathbb{C}[t]$ and let H_x denote the subspace of polynomials vanishing at $x \in \mathbb{C}$. Suppose there exists a basis of $\mathcal{V}$ so that all H_x are coordinate subspaces, and let e be any basis vector which is not in H_0. Then e must belong to H_x for all $x \neq 0$ and therefore $e = 0$, which is a contradiction.

A.1 Regular sequences

Definition A.5. A sequence of commuting endomorphisms $f_1, \dots, f_n$ of a vector space M is called *regular* if for any $k = 1, \dots, n$ and any $m \in M$ satisfying $f_k m \in (f_1, \dots, f_{k-1})M$ we have $m_k \in (f_1, \dots, f_{k-1})M$.

Proposition A.6. *Suppose M is a non-negatively graded vector space, and suppose $f_1, \dots, f_n$ are commuting endomorphisms of M of positive degree. Then the following conditions are equivalent:*

1. *The sequence $f_1, \dots, f_n$ is regular.*
2. *M is a free module over the polynomial ring $\mathbb{C}[f_1, \dots, f_n]$.*
3. *f_i is injective for all i, $f_i M \cap f_j M = f_i f_j M$ for all $i \neq j$ and the subspaces $f_i M \subset M$ generate a distributive lattice.*

Proof. The implication $(i) \Rightarrow (ii)$ is standard. To show that $(ii) \Rightarrow (iii)$ let $(m_i)_{i \in I}$ be a basis of M as a module over the polynomial ring. Then $f_1^{a_1} \cdots f_n^{a_n} m_i$ form a basis of M as a vector space. Clearly, $f_i M$ are coordinate subspaces for this basis. Hence they generate a distributive lattice.

Now let us show that $(iii) \Rightarrow (i)$. Suppose $f_k m \in (f_1, \dots, f_{k-1})M$. This implies

$$f_k m \in f_k M \cap (f_1 M + \cdots + f_{k-1} M) = (f_k M \cap f_1 M) + \cdots + (f_k M \cap f_{k-1} M),$$

and using $f_k M \cap f_i M = f_k f_i M$ and the injectivity of f_k we obtain $m \in f_1 M + \cdots + f_{k-1} M$. □

A.2 Homology

For simplicity of notation, we formulate results in this section for ordinary Borel-Moore homology, but all statements clearly remain valid for equivariant Borel-Moore homology.

Definition A.7. Suppose X is a topological space. A collection of closed subsets $\mathcal{Z}$ is called a *lattice* if for any $Z_1, Z_2 \in \mathcal{Z}$ we have $Z_1 \cap Z_2, Z_1 \cup Z_2 \in \mathcal{Z}$. A lattice of subsets is called *nice* if for any $Z \in \mathcal{Z}$ the map on Borel-Moore homology $H_*(Z) \to H_*(X)$ is injective and for any $Z_1, Z_2 \in \mathcal{Z}$ we have $H_*(Z_1 \cap Z_2) = H_*(Z_1) \cap H_*(Z_2)$, $H_*(Z_1 \cup Z_2) = H_*(Z_1) + H_*(Z_2)$.

Proposition A.8. *Suppose $\mathcal{Z}$ is a collection of closed subsets of X, and suppose for any $Z \in \mathcal{Z}$ the map on Borel-Moore homology $H_*(Z) \to H_*(X)$ is injective, and suppose for any tuple $Z_1, \ldots, Z_m$ the map on Borel-Moore homology*

$$H_*(Z_1 \cap \cdots \cap Z_m) \to H_*(Z_1) \cap \cdots \cap H_*(Z_m)$$

is an isomorphism. Then the following conditions are equivalent:

1. *The lattice generated by $\mathcal{Z}$ is nice.*

2. *The subspaces $H_*(Z) \subset H_*(X)$ for $Z \in \mathcal{Z}$ generate a distributive lattice.*

Proof. $(i) \Rightarrow (ii)$ is evident because if the lattice generated by $\mathcal{Z}$ is nice, then the operations on vector spaces match with the operations on subsets, while on subsets the distributive law $(Z_1 \cup Z_2) \cap Z_3 = (Z_1 \cap Z_3) \cup (Z_2 \cap Z_3)$ is automatic.

Now we show $(ii) \Rightarrow (i)$. Without loss of generality we may assume that $\mathcal{Z}$ is closed under intersections. Then any element in the lattice generated by $\mathcal{Z}$ can be written as a union of elements of $\mathcal{Z}$. Let us show that $H_*(Z_1 \cup \cdots \cup Z_m) = H_*(Z_1) + \cdots H_*(Z_m)$ for any $Z_1, \ldots, Z_m$ by induction on m. Let $A = Z_1 \cup \cdots \cup Z_{m-1}$. We have a long exact sequence

$$\cdots \to H_*(A \cap Z_m) \to H_*(A) \oplus H_*(Z_m) \to H_*(A \cup Z_m) \to \cdots \tag{60}$$

By the induction assumption, the assumptions on $\mathcal{Z}$, and the distributivity assumption we have

$$H_*(A \cap Z_{m-1}) = H_*((Z_1 \cap Z_m) \cup \cdots \cup (Z_{m-1} \cap Z_m))$$

$$= H_*(Z_1 \cap Z_m) + \cdots + H_*(Z_{m-1} \cap Z_m)$$

$$= (H_*(Z_1) \cap H_*(Z_m)) + \cdots + (H_*(Z_{m-1}) \cap H_*(Z_m))$$

$$= (H_*(Z_1) + \cdots + H_*(Z_{m-1})) \cap H_*(Z_m) = H_*(A) \cap H_*(Z_m).$$

Thus in particular the second arrow in (60) is injective and the long exact sequence splits into short exact sequences. This implies that $H_*(A \cup Z_m)$ is the quotient $H_*(A) \oplus H_*(Z_m)/H_*(A) \cap H_*(Z_m)$, which is isomorphic to the sum $H_*(A) + H_*(Z_m)$, so the induction step is proved.

Now suppose $A = \bigcup_i Z_i$, $B = \bigcup_i Z_i'$ are arbitrary elements of the lattice generated by $\mathcal{Z}$. We have

$$H_*(A) = \sum_i H_*(Z_i), \quad H_*(B) = \sum_i H_*(Z_i'),$$

and therefore

$$H_*(A) + H_*(B) = \sum_i H_*(Z_i) + \sum_i H_*(Z_i') = H_*(A \cup B),$$

$$H_*(A) \cap H_*(B) = \sum_{i,j} H_*(Z_i) \cap H_*(Z_j') = H_* \left(\bigcup_{i,j} Z_i \cap Z_j' \right) = H_*(A \cap B).$$

□

A useful tool for constructing nice lattices are affine pavings.

Proposition A.9. *Suppose X is paved by finitely many sets Z_α indexed by $\alpha \in \Lambda$ where Λ is a poset such that for any lower set $A \subset \Lambda$ the union $Z_A := \bigcup_{\alpha \in A} Z_\alpha$ is closed. Suppose for any $\alpha \in \Lambda$ the odd homologies $H_{2i+1}(Z_\alpha)$ vanish. Then the subsets of the form Z_A form a nice lattice of sets.*

Proof. Applying long exact sequences and the vanishing of odd homology we see that the odd homologies of Z_A vanish for any finite lower set A and for any $A \subset B$ the map $H_*(Z_A) \to H_*(Z_B)$ is injective. Now consider arbitrary lower sets A, B. We have a short exact sequence

$$0 \to H_*(Z_A \cap Z_B) \to H_*(Z_A) \oplus H_*(Z_B) \to H_*(Z_A \cup Z_B) \to 0.$$

Since $H_*(Z_A \cup Z_B)$ embeds into $H_*(X)$ we obtain that $H_*(Z_A \cup Z_B) = H_*(Z_A) + H_*(Z_B)$ and $H_*(Z_A \cap Z_B) = H_*(Z_A) \cap H_*(Z_B)$. □

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