1. Find the inverse of the following matrices

(a) \[
\begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & \pi & 0 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 0 & -7
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
0 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 \\
0 & 5 & 0 & 0 \\
-4 & 0 & 0 & 0
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
3 & 4 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

2. We call a matrix invertible if it has an inverse and a matrix is called singular if it has no inverse. The sum \(A + B\) of two square matrices \(A\) and \(B\) is defined component wise, hence for example

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix} + \begin{bmatrix}
\pi & -1 \\
0 & 7
\end{bmatrix} = \begin{bmatrix}
1 + \pi & 1 \\
3 & 11
\end{bmatrix}
\]

(a) Give an example of invertible matrices \(A\) and \(B\) with \(A + B\) singular.

(b) Give an example of singular matrices \(A\) and \(B\) such that \(A + B\) is invertible.

3. Show that a matrix of the form

\[
\begin{bmatrix}
* & * & 0 & * \\
* & * & 0 & * \\
* & * & 0 & * \\
* & * & 0 & *
\end{bmatrix}
\]

has no inverse.

4. For

\[
A = \begin{bmatrix}
2 & 2 & 1 \\
4 & 0 & -3 \\
-10 & 1 & 1
\end{bmatrix}
\]

find a factorization \(A = L \cdot U\) where \(L\) is lower triangular and \(U\) is upper triangular.
5. Show that

\[ A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \]

has no \( A = L \cdot U \) factorization where \( L \) is lower triangular and \( U \) is upper triangular: Suppose for contradiction that

\[ A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} \]

and show that there are no choices of \( a, b, c, x, y, z \) such that this equation holds.

6. Show that

\[ A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]

has two different factorizations of the form \( A = L \cdot U \) where \( L \) is lower triangular with 1’s on the diagonal and \( U \) is upper triangular.

7. Recall that a subset \( V \) of \( \mathbb{R}^n \) is a subspace if two conditions hold:
   - Condition (I): For all \( \mathbf{x}, \mathbf{y} \) in \( V \) one has \( \mathbf{x} + \mathbf{y} \) in \( V \).
   - Condition (II): For all \( \mathbf{x} \) in \( V \) and all scalars \( c \) one has \( c \cdot \mathbf{x} \) in \( V \).

   (a) Find a subset \( V \) of \( \mathbb{R}^2 \) such that condition (I) holds but condition (II) fails.
   (b) Find a subset \( V \) of \( \mathbb{R}^2 \) such that condition (II) holds but condition (I) fails.

8. Decide, with justification if the following subsets of \( \mathbb{R}^3 \) are in fact subspaces of \( \mathbb{R}^3 \):
   (a) \[ V = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right\} \text{ with } b_1 \cdot b_2 + b_3 = 0 \]
   (b) \[ V = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right\} \text{ with } 2 \cdot b_1 - \pi \cdot b_2 + b_3 = 0 \]
   (c) \[ V = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right\} \text{ with } b_1 \cdot b_3 \leq 10 \]