1) Let $G$ be a group. A commutator in $G$ is an element of the form $aba^{-1}b^{-1}$ with $a,b \in G$. Let $G^c$ be the subgroup generated by the commutators, called the commutator subgroup. Show that $G^c$ is normal in $G$. Challenge: Show that any homomorphism $\phi$ of $G$ into an abelian group factors through $G/G^c$, meaning that there exists a map $f$ such that $\phi = f \circ \pi$ where $\pi : G/G^c$ is the canonical morphism.

2) Let $H,K$ be subgroups of a finite group $G$. Assume $K \subseteq N_H$. Show that $\#(HK) = \#(H)\#(K)/\#(H \cap K)$.

3) Let $G$ be a group and let $H$ be a subgroup of finite index. Prove that there is only a finite number of right cosets of $H$, and that the number of right cosets is equal to the number of left cosets.

4) a) Let $H,N$ be normal subgroups of a finite group $G$. Assume that the orders of $H,N$ are relatively prime. Prove that $xy = yx$ for all $x \in H$ and $y \in N$, and that $H \times N \cong HN$.

b) Let $H_1,\ldots,H_r$ be normal subgroups of $G$ such that the order of $H_i$ is relatively prime to the order of $H_j$ for $i \neq j$. Prove that $H_1 \times \ldots \times H_r \cong H_1 \cdots H_r$.

5) Let $p$ be a prime and let $G$ be of order $p^n$ where $n > 0$ with center $Z(G) \neq 1$. Show that $G$ has a chain of subgroups

$$G = G_0 > G_1 > G_2 > \cdots > G_n = 1$$

such that $G_i$ is normal in $G$ and $[G : G_i] = p^i$ for all $i$. What are the composition factors of $G$?

Hint: Use the fact that $Z(G) \neq 1$ to produce an element $x \in Z(G)$ of order $p$. Prove by induction, considering the quotient group $G/\langle x \rangle$.

6) The dihedral group $D_8$ containing 8 elements has seven different composition series. Find all of them.

7) a) Show that an abelian group has a composition series if and only if it is finite.

b) Let $F$ be a field and let $GL_n(F)$ denote the group of $n \times n$ invertible matrices with entries in $F$ (the group operation is matrix multiplication). Show that $GL_n(F)$ has a composition series if and only if $F$ is finite.