1) Prove that the group of inner automorphisms of a group $G$ is normal in $\text{Aut}(G)$.

2) Let $G$ be a group such that $\text{Aut}(G)$ is cyclic. Prove that $G$ is abelian.

3) Let $P$ be a $p$-group. Let $A$ be a normal subgroup of order $p$. Prove that $A$ is contained in the center of $P$. In case we haven’t gotten to this in class yet, given a prime $p$, a group of order $p^n$ for some $n \geq 0$ is called a $p$-group.

4) A nontrivial fact is that $[\text{Aut}(S_6) : \text{Inn}(S_6)] = 2$. In this problem we will show that $[\text{Aut}(S_6) : \text{Inn}(S_6)] \leq 2$ and that $\text{Aut}(S_n) = \text{Inn}(S_n)$ for all $n \geq 3$ and $n \neq 6$. Throughout this problem, $n \geq 3$.

   a) Let $C$ be the conjugacy class of any transposition in $S_n$, and let $C'$ be the conjugacy class of any element of order 2 in $S_n$ which is not a transposition. Show that $|C| \neq |C'|$ unless $n = 6$ and $C'$ is the conjugacy class of a product of three disjoint transpositions. Deduce that $\text{Aut}(S_6)$ has a subgroup of index at most 2 which sends transpositions to transpositions, and that any automorphism of $S_n$ where $n \neq 6$ sends transpositions to transpositions.

   b) Show that any automorphism that sends transpositions to transpositions is inner, and deduce that $\text{Aut}(S_n) = \text{Inn}(S_n)$ if $n \neq 6$ and $[\text{Aut}(S_6) : \text{Inn}(S_6)] \leq 2$ from part (a).

5) Show that a simple group whose order is $\geq r!$ cannot have a proper nontrivial subgroup of index $r$.

6) In this problem, you are allowed to assume that $A_n$ is simple for $n \geq 5$. Show that $S_n$ has no proper subgroups of index $< n$ other than $A_n$ for $n \geq 5$.

7) A chief series of a group $G$ is a series of subgroups $G = G_0 > G_1 > \cdots > G_r = 0$ such that $G_i \triangleleft G$ for all $1 \leq i \leq r$ and such that no normal subgroup of $G$ is contained properly between any two terms in the series. The factors $G_i/G_{i+1}$ in this series are called chief factors.

   a) We say that $N$ is a minimal normal subgroup of a group $G$ if $1 \neq N \triangleleft G$ such that no nontrivial normal subgroup of $G$ is properly contained in $N$. Show that every finite group $G$ has a chief series
and that any minimal normal subgroup $N$ of a finite group $G$ is a chief factor in some chief series of $G$. (In fact, chief factors, much like composition factors, are “unique” to a group which admits a chief series, and if you’d like you can formulate and prove for yourself the analog of Jordan-Hölder for chief series)

b) Show that any group having a composition series also has a chief series.

8) A central series of a group $G$ is a series of subgroups

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_r = 0$$

such that $G_i \trianglelefteq G$ for all $1 \leq i \leq r$ and such that each quotient $G_i/G_{i+1}$ is contained in the center of $G/G_{i+1}$. A group $G$ is called nilpotent if it admits a central series.

a) Show that nilpotent groups are solvable.

b) Give an example of a solvable group which is not nilpotent.