Math 250A Homework 7, due 12/6/2019

In the first few problems, we will walk through the proof of the fundamental theorem of algebra. First, a definition:

Let $K/F$ be a finite extension. The **normal closure** of $K/F$ is the splitting field over $F$ of

\[ \{ \text{minimal polynomial of } \alpha \text{ over } F \mid \alpha \in K \} \]

1. Let $K/F$ be a finite extension and let $N$ be the normal closure of $K/F$.
   a) Show that $N$ is a normal extension of $F$ containing $K$ and whenever $M$ is a normal extension of $F$ with $K \subseteq M \subseteq N$ then $M = N$.
   b) Show that if $K = F(\alpha_1, \ldots, \alpha_r)$ then $N$ is the splitting field of the set of minimal polynomials of $\alpha_i$'s over $F$.
   c) Show that $N/F$ is a finite extension.
   d) Show that if $K/F$ is separable then $N/F$ is Galois.
   e) Pick a favorite finite extension of some field $F$ that is not normal and find its normal closure.

2. a) Show that if $K/\mathbb{R}$ is a finite extension with odd degree, then $K = \mathbb{R}$.
   b) Show that if $K/\mathbb{C}$ is a finite extension such that $[K : \mathbb{C}] \leq 2$ then $K = \mathbb{C}$.

3) Let $K/\mathbb{C}$ be a finite extension. Suppose $N$ is the normal closure of $K/\mathbb{R}$. Suppose $N \neq \mathbb{C}$.
   a) Given the above, show $2|\text{Gal}(N/\mathbb{R})|$ and that there is a nontrivial extension of odd degree $L/\mathbb{R}$ contained in $N$.
   b) Show that both $\text{Gal}(N/\mathbb{R})$ and $\text{Gal}(N/\mathbb{C})$ are nontrivial 2-groups.
   c) Show that $\text{Gal}(N/\mathbb{C})$ has an index 2 subgroup $H$ and determine the degree of $\mathcal{F}(H)/\mathbb{C}$. Conclude that $\mathbb{C}$ is algebraically closed.

4. Determine the Galois group of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$. Find the intermediate fields and corresponding subgroups.