Todd class of the permutohedral variety.

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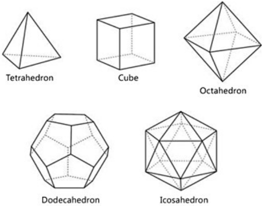
What is a polytope?

Definition

A polytope is the convex hull of finitely many points.



A polytope has *dimension*. These are the regular polytopes in dimension 3.

Integer points

We are mainly interested on the **integer points** for several reasons.

- There are many problems in discrete optimization in which the feasible solutions are required to be integers (e.g. traveling salesman problem).
- Given a (lattice) polytope P one can construct a projective toric variety X_P together with an ample divisor D_P such that $\chi(X_P, D_P) = h^0(X_P, D_P) = |P \cap \mathbb{Z}^n|$, so we can use well developed methods in algebraic geometry for computing global sections.

Counting

We will now try to describe formulas for the number of points. We will restrict ourselves to polytopes whose vertices are **integer** points. Informally, we want our polytopes to be *snapped to the grid* $\mathbb{Z}^n$.

Theorem (Pick 1899)

The area of a polygon is equal to the number of interior points, plus half the number of boundary points minus 1.

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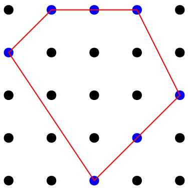
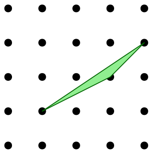


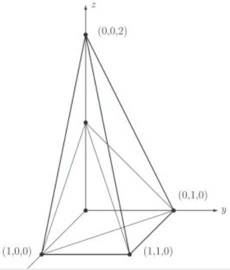
$$\frac{19}{2} = 7 + \frac{7}{2} - 1$$

Gist of the proof

The main point is that triangles with no other integer point other than the vertices (empty simplices) have area 1/2.



The previous observation fails already in dimension 3.



Empty simplices

Empty simplices are elusive objects.

1. Dimension 2: Just one (up to unimodular equivalence).
2. Dimension 3: Infinite but classified (White 1964).
3. Dimension 4: Several infinite families plus some outliers. Now fully classified (Iglesias-Santos 2018).
4. Dimension > 4: Very little is known.

Towards a higher Pick?

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The generalization we are looking for **cannot** depend just on the integer points! What we are for instead is this

McMullen Formula

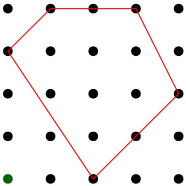
$$|P| = \sum_{F \subset P} \alpha(F, P)\text{relvol}(F)$$

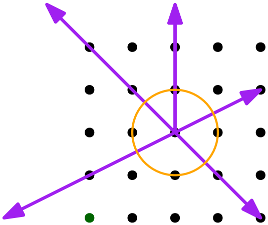
.

The $\alpha(F, P)$ are local in the sense that they depend only on the normal fan of F. Informally the values depend on what's nearby F. Peter McMullen proved the existence of such α in a nonconstructive and nonunique way. By now we have at least three different constructions.

Dependence on the normal fan

For each face of a polytope, we can associate a cone, the *normal cone*. The union is the normal fan.



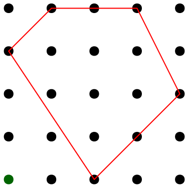


The normal cone of a face G is the cone whose rays correspond the outer normal to the facets F with $G \subset F$.

Pick revisited

Let's rewrite Pick from $A = I + \frac{B}{2} - 1$ to

$$|P| = A(P) + \frac{B(P)}{2} + 1$$

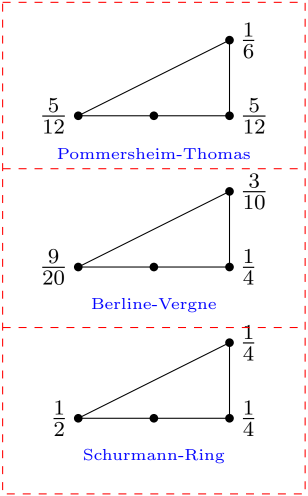


In this case $14 = 1 \cdot \frac{19}{2} + \frac{1}{2}(2 + 1 + 2 + 1 + 1) + \mathbf{1}$.

Note:

The $+1$ must know come from contributions from all the vertices. But how?

Example



McMullen Formula:

$$|P \cap \mathbb{Z}| = (\text{Area of P}) + \frac{1}{2}(\text{Perimeter of P}) + 1.$$

The way one gets the +1 is different.

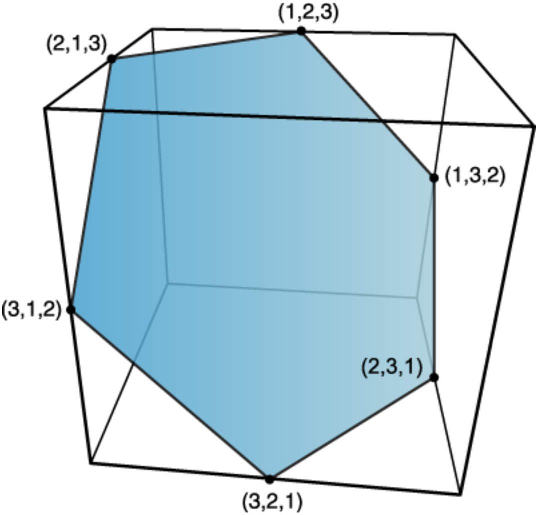
Different constructions

- Pommersheim-Thomas 2004: Compute the Todd class of the associated toric variety.
- Berline-Vergne 2007: Coming from Euler Mclaurin formulas.
- Schurmann-Ring 2017: Several volume computation of possibly non-convex bodies.

"We're living in a world that come with plan B
Cause plan A never relay a guarantee
And plan C never could say just what it was."

–Kendrick Lamar.

Main Example: Regular permutohedron



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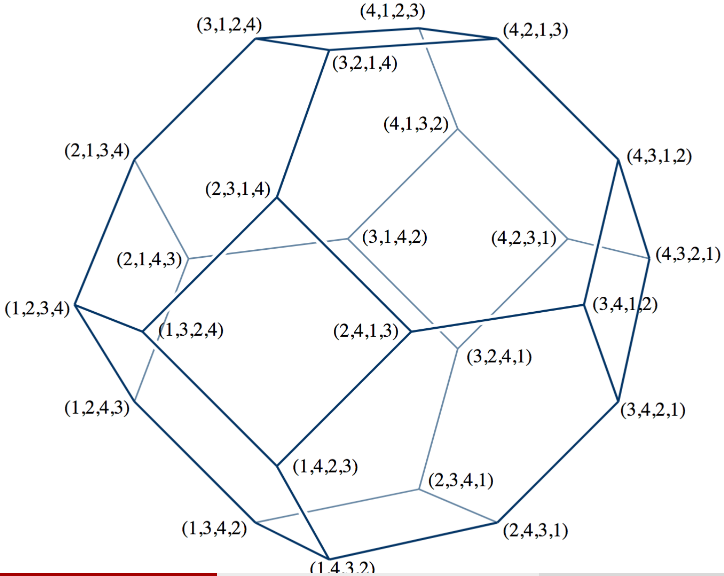

Generalized permutohedra

The normal fan of Π_n is the **Braid Fan** of type A and we denote it $\mathcal{A}_n$.

Definition

A **generalized permutohedron** is a polytope P such that the fan $\mathcal{A}$ refines the normal of P.

Alternatively,

Equivalence

A **generalized permutohedron** is a polytope P such that every edge is parallel to $e_i - e_j$ for some i, j. In other words, the edges are parallel to type A roots.

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Conjecture(C.-Liu 2016)

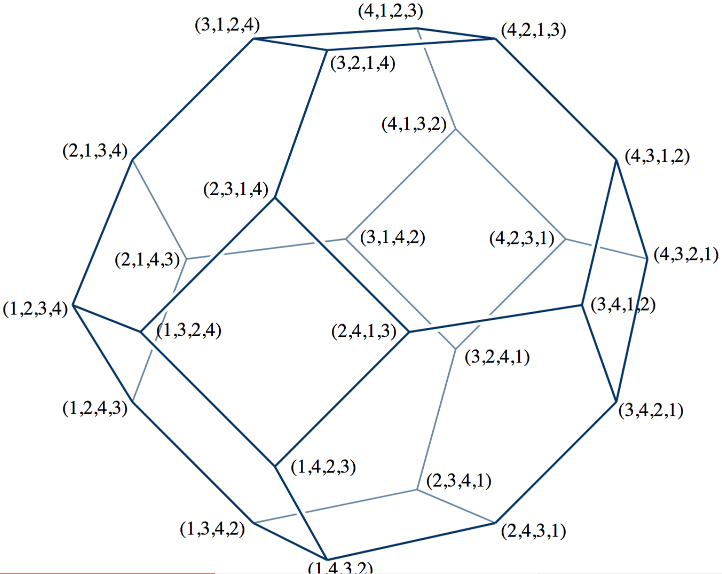
Under certain symmetry restrictions **and the valuation property**, there is a unique construction.

Computations

- There is just one 3 dimensional face, with $\alpha = 1$ and volume $4^{4-2} = 16$ which contributes **16**
- There are six 2 dimensional faces with volume 1 and eight with volume 3. The α value is $1/2$ for all these faces so we get a contribution of $6\frac{1}{2}1 \cdot +8\frac{1}{2}3\cdot = \mathbf{15}$
- Two types of edges, both with volume 1. There are 24 short edges with value $11/72$ and 12 long edges with value $14/72$, for a contribution of $24\frac{11}{72} + 12\frac{14}{72} = \mathbf{6}$
- There are 24 vertices, all with value $1/24$ with a contribution of **1**

For a total number of points of $38 = 16 + 15 + 6 + 1$.

Regular permutohedron



Results on the permutohedron.

The normal fan of the regular permutohedron is called the **braid fan.**

Results (C.-Liu 2016, 2019)

- All edges are α-positive.
- Formula for such things involving *mixed Ehrhart coefficients* of hypersimplices.
- Formula using *any* expression for the Todd class of the permutohedral variety.

We conjectured that all α values were positive in this case.

Ehrhart Theory

Our motivation came from trying to solve another conjecture.

Ehrhart Polynomial

Let P be a d-dimensional lattice polytope. There exists a polynomial $\mathrm{Ehr}_P(t) \in \mathbb{Q}[t]$ such that $\mathrm{Ehr}_P(n) = \mathrm{Lat}(nP)$ for $n \in \mathbb{N}$.

If $\mathrm{Ehr}_P(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t^1 + a_0$ with $a_i > 0$, we call P**Ehrhart positive.**

Conjecture(De Loera, Haws, Koeppe)

Matroid polytopes are Ehrhart positive.

Conjectures

Raising the bet (C.-Liu 2016)

(integral) Generalized permutohedra are Ehrhart positive.

Refined conjecture

The α values in the braid fan are positive.

This is *strictly stronger* than the previous one due to an example we found in joint work with B.Nill and A.Paffenholz in 2017.

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Spoiler Alert

The refined conjecture is false.

Onward to Todd classes

Let X be the toric variety associated to the regular permutohedron. This is called the **permutohedral variety**. The Todd class $\mathrm{Td}(X)$ is an element in the Chow ring of X. As such it can be written as a $\mathbb{Q}$-linear combination of the toric invariant cycles $[V(\sigma)]$ (for each σ cone in the normal fan):

$$\mathrm{Td}(X) = \sum_{\sigma \in \Sigma} r(\sigma)\ [V(\sigma)], \quad r(\sigma) \in \mathbb{Q}. \tag{1}$$

This is relevant because of the Riemann-Roch-Hirzebruch theorem that says $\chi(X, D) = \int_X \mathrm{ch}(D)\,\mathrm{Td}(X)$.

In other words

We can translate from

$$\mathrm{Td}(X) = \sum_{\sigma \in \Sigma} r(\sigma)\ [V(\sigma)] \tag{2}$$

to

$$|P| = \sum_{F \subset P} r(\sigma)\mathrm{relvol}(F) \tag{3}$$

(actually not just for P being the permutohedron but for any polytope with same normal fan)

In our concrete case

The Chow ring of the permutohedral variety X_d can be presented as

$$A_d \cong R_d/(I_1 + I_2) \tag{4}$$

where $R_d = k[x_S : S \subset [d+1]]$,

$I_1 = \langle x_S x_{S'} :\ \text{for } S, S' \text{ incomparable}\rangle, \quad I_2 = \langle \ell_a - \ell_b :\ \text{for all } a, b \in [d+1]$

and $\ell_i := \sum_{S \ni i} x_S$.

Todd class

The **Todd class of** X_d is the element of A_d defined as

$$\mathrm{Td}(X_d) := \prod_S \left(\frac{x_S}{1 - e^{-x_S}} \right), \tag{5}$$

which is an element of A_d by expanding each parenthesis on the right hand side as

$$\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x^{2i} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} + \cdots . \tag{6}$$

Main idea: Expand and write everything in terms of **square-free** monomials in a **symmetric** way.

Main result

Theorem (C.-Liu 2019)

A combinatorial formula for $\alpha(P, F)$ whenever P is a regular permutohedron and F a face.

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Note: This allow us to disprove ourselves.

Corollary

The Todd class of the permutohedral variety is not effective. That is, there is no way to write it as a combination of nonnegative cycles.

Example

Formula for arbitrary 4-dimesional cones (or codimension 4 faces) comes from

	$\frac{1}{16}$		$-\frac{1}{48}\frac{d+1-s_4}{d+1-s_3}$		$\frac{1}{144}\frac{s_1}{s_2}\frac{s_3-s_2}{s_4-s_2}$
	$-\frac{1}{48}\frac{s_3-s_2}{s_3-s_1}$		$-\frac{1}{48}\frac{s_3-s_2}{s_4-s_2}$		$\frac{1}{720}\frac{s_3-s_2}{s_3-s_1}\frac{s_4-s_3}{s_4-s_1}\frac{d+1-s_4}{d+1-s_1}$
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Conclusion

The following is still open

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(integral) Generalized permutohedra are Ehrhart positive.

Recently C.-Liu and independently Jochemko-Ravichandran proved that the linear term is always positive. Additionally, Ferroni proved that hypersimplices are Erhart positive. Even more recently, Ferroni conjecture a new plan to prove it for matroid polytopes.

Conclusion

Mystery remains open.

Final, final, no va mas.

The End