

# Poisson slices and Hessenberg varieties

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# Overview

- 1 Poisson slices
- 2 The wonderful compactification
- 3 Hessenberg varieties

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## Definition

The Poisson variety  $\mu^{-1}(\mathcal{S}_\tau)$  is called a *Poisson slice*.

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- (iv) If  $X$  is log symplectic, then each irreducible component of  $\mu^{-1}(\mathcal{S}_\tau)$  is a log symplectic subvariety of  $X$ .

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In what follows, we fix a principal  $\mathfrak{sl}_2$ -triple  $\tau = (e, h, f)$  and set  $\mathcal{S} := \mathcal{S}_\tau$ .

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- Consider the principal  $\mathfrak{sl}_2$ -triple  $((e, -e), (h, h), (f, -f))$  and associated Slodowy slice  $\mathcal{S} \times (-\mathcal{S}) \subseteq \mathfrak{g} \oplus \mathfrak{g}$ .

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- $\mu^{-1}(\mathcal{S} \times (-\mathcal{S})) = \{(g, \xi) \in G \times \mathfrak{g} : \xi \in \mathcal{S} \text{ and } g \in G_\xi\}$
- This symplectic subvariety of  $T^*G$  is called the *universal centralizer* and denoted  $\mathcal{Z}_\mathfrak{g}$ .



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- $\mu_R^{-1}(-\mathcal{S}) = G \times \mathcal{S}$ , a symplectic subvariety of  $T^*G$

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## Theorem (De Concini–Procesi)

The projective variety  $\overline{G}$  is smooth, and  $D := \overline{G} \setminus G$  is a normal crossings divisor in  $\overline{G}$ .

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- (ii)  $T^*G$  is the unique open dense symplectic leaf in  $T^*\overline{G}(\log(D))$ .
- (iii) The action of  $G \times G$  on  $\overline{G}$  canonically lifts to a Hamiltonian  $(G \times G)$ -action on  $T^*\overline{G}(\log(D))$ , and there is an explicit moment map  $\overline{\mu} = (\overline{\mu}_L, \overline{\mu}_R) : T^*\overline{G}(\log(D)) \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ .

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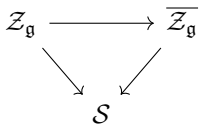
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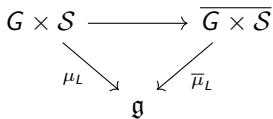
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- Consider the two diagrams



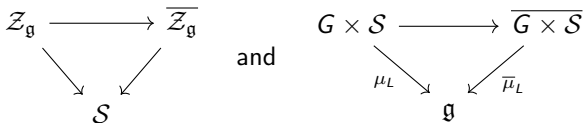
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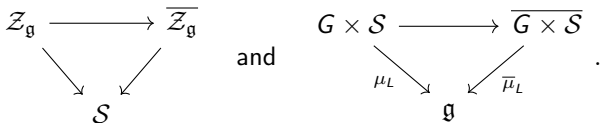


- The former is obtained by pulling the latter back along the inclusion  $S \rightarrow \mathfrak{g}$ . In other words, the former is the restriction of the latter to the Poisson slices

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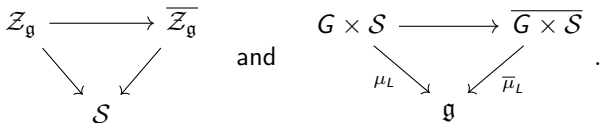
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- Each diagram is a fibrewise compactification.
- The compactifying spaces  $\overline{\mathcal{Z}}_{\mathfrak{g}}$  and  $\overline{G \times \mathcal{S}}$  are related to *Hessenberg varieties*.

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- (ii) The action  $G \curvearrowright G \times_B \mathfrak{m}$  is Hamiltonian with moment map

$$\nu : G \times_B \mathfrak{m} \rightarrow \mathfrak{g}, \quad [g, x] \mapsto \text{Ad}_g(x).$$

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- (iii) If  $x \in \mathfrak{g}$  is regular and nilpotent, then  $\nu^{-1}(x)$  is isomorphic to the Peterson variety.

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There is a canonical isomorphism

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If  $x \in \mathfrak{g}$ , then there is a canonical closed embedding  $\nu^{-1}(x) \rightarrow \overline{G}$ .