Poisson slices and Hessenberg varieties

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Overview

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Definition

The Poisson variety $\mu^{-1}(\mathcal{S}_\tau)$ is called a *Poisson slice*.

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(iii) If X is symplectic, then $\mu^{-1}(\mathcal{S}_\tau)$ is a symplectic subvariety of X.

(iv) If X is log symplectic, then each irreducible component of $\mu^{-1}(\mathcal{S}_\tau)$ is a log symplectic subvariety of X.

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In what follows, we fix a principal $\mathfrak{sl}_2$-triple $\tau = (e, h, f)$ and set $\mathcal{S} := \mathcal{S}_\tau$.

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- Consider the principal $\mathfrak{sl}_2$-triple $((e, -e), (h, h), (f, -f))$ and associated Slodowy slice $\mathcal{S} \times (-\mathcal{S}) \subseteq \mathfrak{g} \oplus \mathfrak{g}$.

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- $\mu^{-1}(\mathcal{S} \times (-\mathcal{S})) = \{(g, \xi) \in G \times \mathfrak{g} : \xi \in \mathcal{S} \text{ and } g \in G_\xi\}$
- This symplectic subvariety of T^*G is called the *universal centralizer* and denoted $\mathcal{Z}_\mathfrak{g}$.

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Theorem (De Concini–Procesi)

The projective variety $\overline{G}$ is smooth, and $D := \overline{G} \setminus G$ is a normal crossings divisor in $\overline{G}$.

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(iii) The action of $G \times G$ on $\overline{G}$ canonically lifts to a Hamiltonian $(G \times G)$-action on $T^*\overline{G}(\log(D))$, and there is an explicit moment map $\overline{\mu} = (\overline{\mu}_L, \overline{\mu}_R) : T^*\overline{G}(\log(D)) \to \mathfrak{g} \oplus \mathfrak{g}$.

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- The former is obtained by pulling the latter back along the inclusion $\mathcal{S} \to \mathfrak{g}$. In other words, the former is the restriction of the latter to the Poisson slices

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- The former is obtained by pulling the latter back along the inclusion $\mathcal{S} \to \mathfrak{g}$. In other words, the former is the restriction of the latter to the Poisson slices

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- Each diagram is a fibrewise compactification.
- The compactifying spaces $\overline{\mathcal{Z}_{\mathfrak{g}}}$ and $\overline{G \times \mathcal{S}}$ are related to *Hessenberg varieties*.

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(i) $G \times_B \mathfrak{m}$ is naturally a Poisson variety.

(ii) The action $G \circlearrowright G \times_B \mathfrak{m}$ is Hamiltonian with moment map

$$\nu : G \times_B \mathfrak{m} \to \mathfrak{g}, \quad [g, x] \mapsto \mathrm{Ad}_g(x).$$

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(iii) If $x \in \mathfrak{g}$ is regular and nilpotent, then $\nu^{-1}(x)$ is isomorphic to the Peterson variety.

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Corollary

If $x \in \mathcal{S}$, then there is a canonical variety isomorphism $\nu^{-1}(x) \cong \overline{G_x}$.

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Corollary

If $x \in \mathfrak{g}$, then there is a canonical closed embedding $\nu^{-1}(x) \to \overline{G}$.