Combinatorics and real lifts of bitangents to tropical plane quartics

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Joint work with Hannah Markwig (U. Tuebingen, Germany)

(arXiv:2004.10891)

Algebraic Geometry Seminar UC Davis

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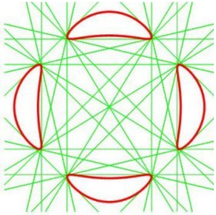

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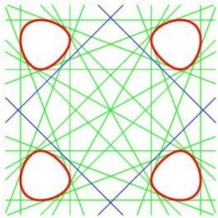

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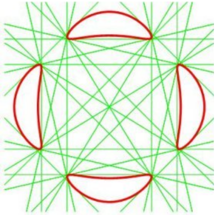

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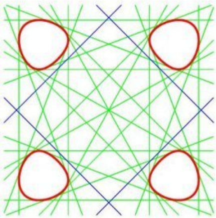

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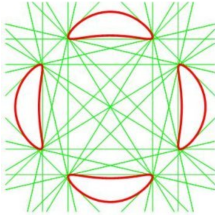

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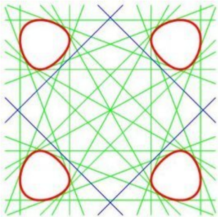

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GOAL: Use tropical geometry to find bitangents over $\mathbb{C}\{\{t\}\}$ and $\mathbb{R}\{\{t\}\}$.

28 bitangent lines to sm. plane quartics over $\mathbb{K}=\overline{\mathbb{C}(\!(t)\!)}$.

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Len-Jensen (2018): Each class *always* lifts to 4 classical bitangents.

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Question 2: What is a tropical bitangent class?
Answer: Continuous translations preserving bitangency properties.

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Theorem: There are 28 classical bitangents to sm. plane quartics over $\mathbb{K}$ **but** 7 tropical bitangent classes to their smooth tropicalizations in $\mathbb{R}^2$.

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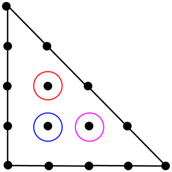
Trop. sm. quartic = dual to unimodular triangulation of Δ_2 of side length 4.



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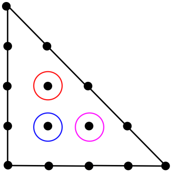

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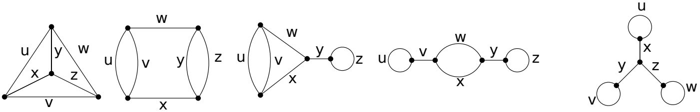

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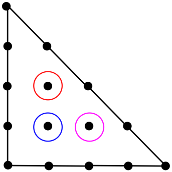

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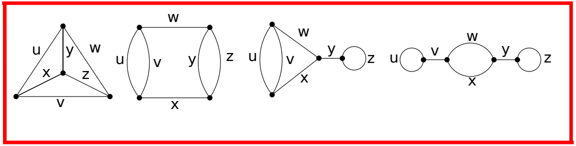

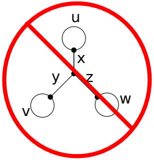

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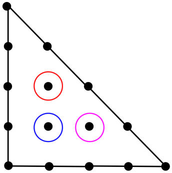

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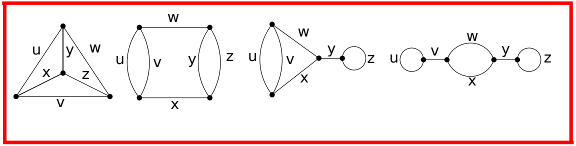

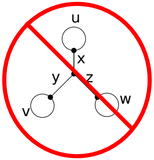

Brodsky-Joswig-Morrison-Sturmfels (2015): Newton subdivisions give linear restrictions on the lengths u, v, w, x, y, z of the edges.

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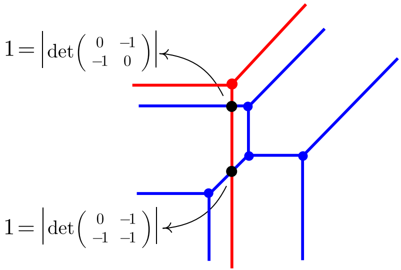
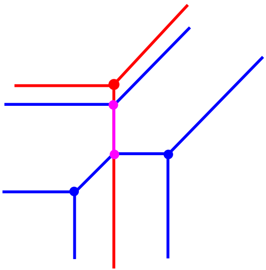


Proper intersection at 2 pts

vs.



Tangent Line

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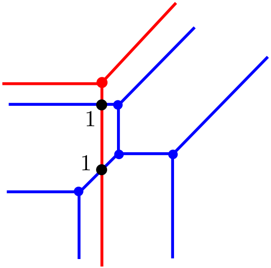

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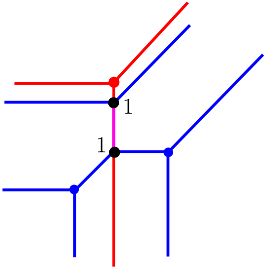

Stable intersection at 2 pts

Non-general case: Replace usual intersection with stable intersection.

$$C_1 \cap_{st} C_2 := \lim_{\underline{\varepsilon} \to (0,0)} C_1 \cap (C_2 + \underline{\varepsilon}).$$

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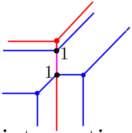

Stable intersection (2 pts)

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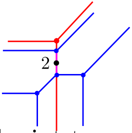

Midpoint tangency

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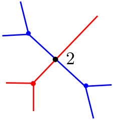

Proper intersection

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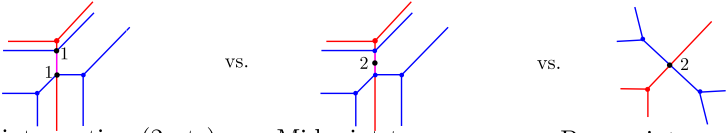

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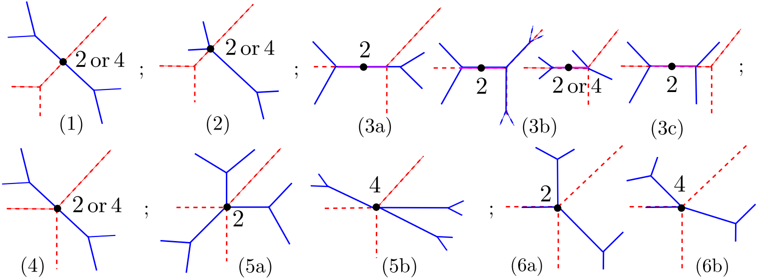

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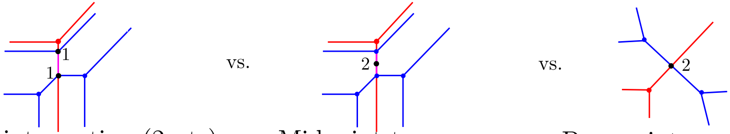

Stable intersection (2 pts) vs. Midpoint tangency vs. Proper intersection

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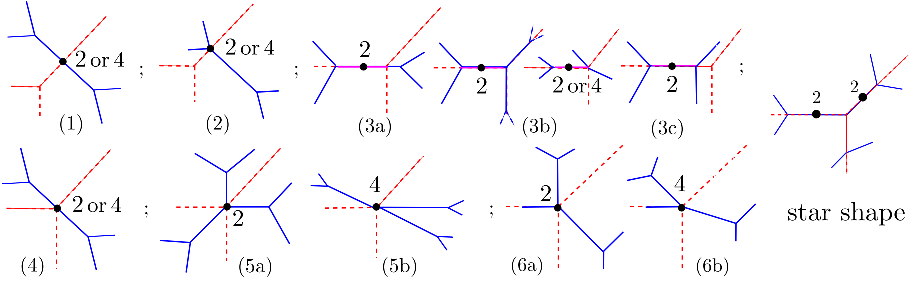

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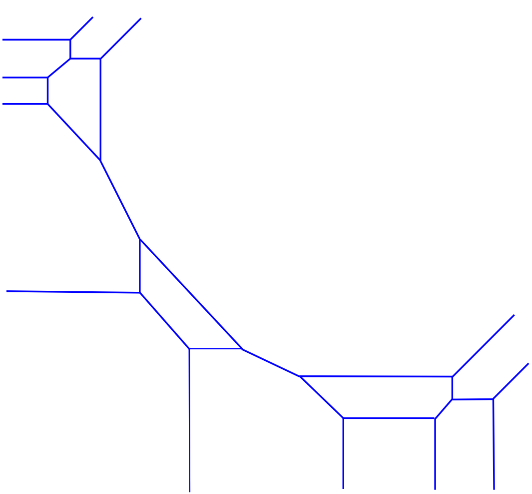

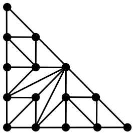

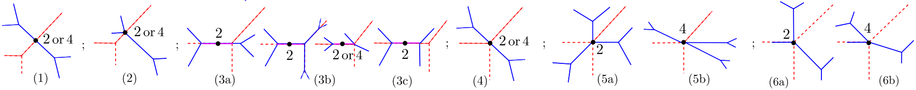

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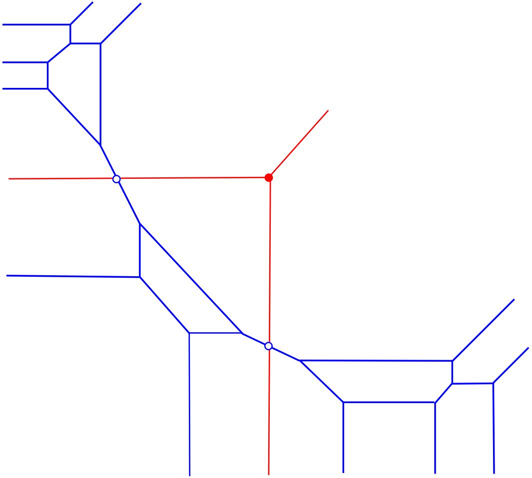

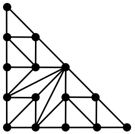

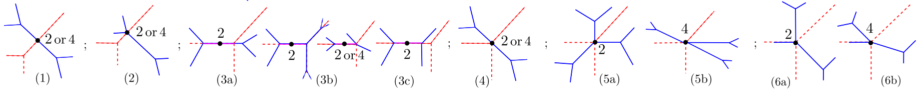

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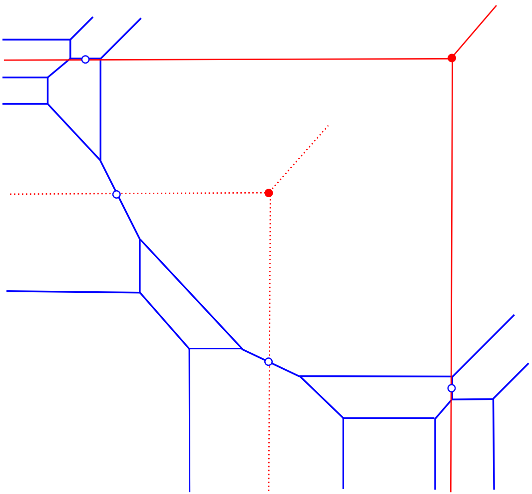

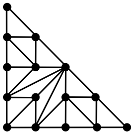

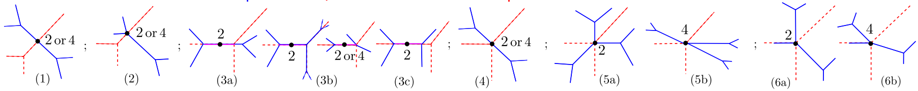

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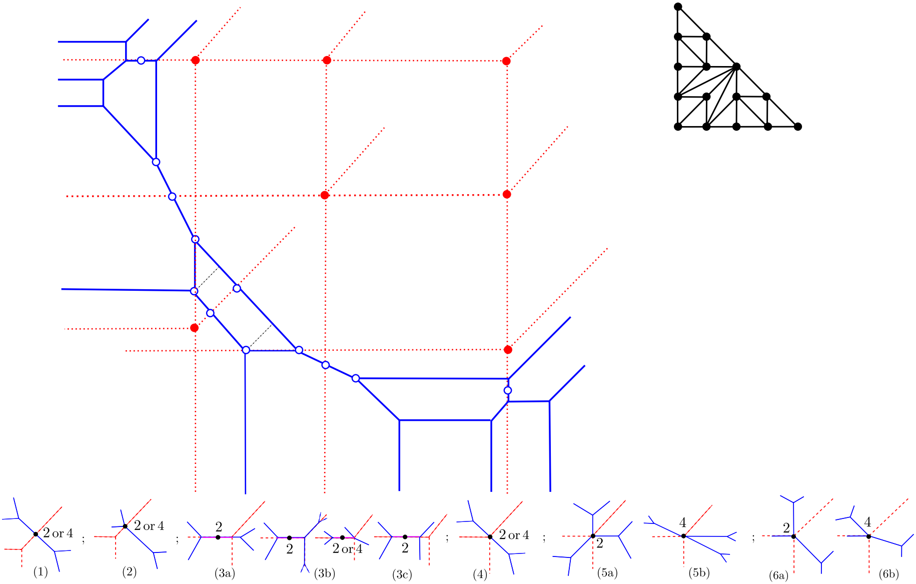

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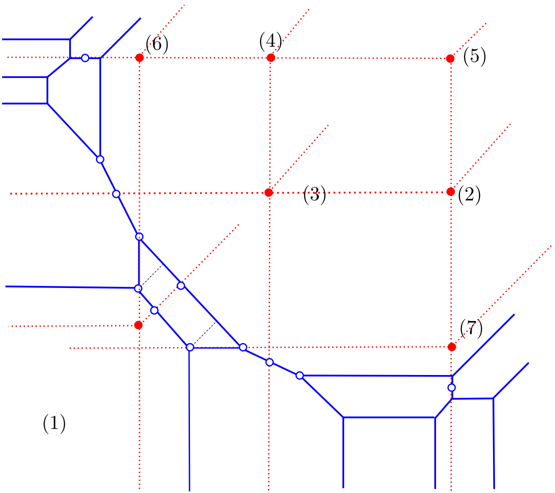

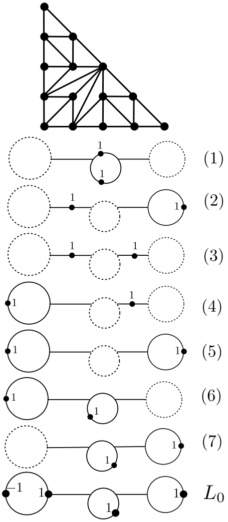

Zharkov (2010): Trop. theta char on a metric graph $\Gamma \leftrightarrow H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$.
$2\theta_i \sim K_\Gamma = \sum_{x\in\Gamma}(\text{val}(x) - 2)x$; L_0 non-effective $\leftrightarrow$ **0**; $2^{b_1(\Gamma)-1}$ effectives.

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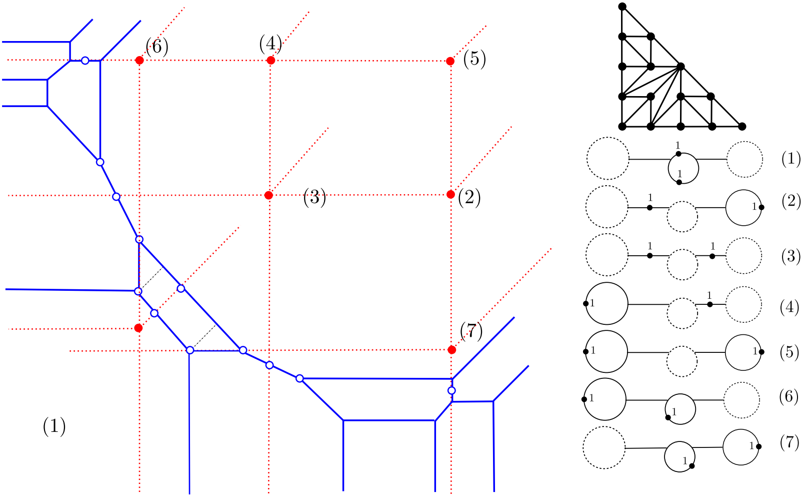

[BLMPR '16]: 7 effective trop. theta characteristics on **skeleton** of tropical sm. quartic Γ in $\mathbb{R}^2$ produce 7 tropical bitangent lines Λ to Γ.

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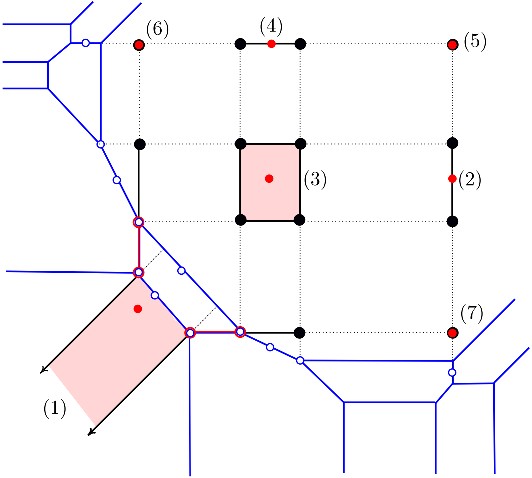
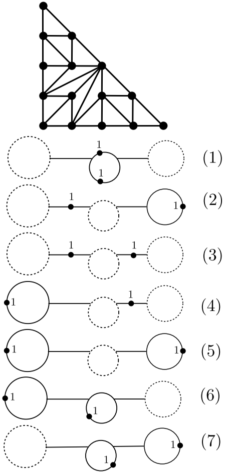
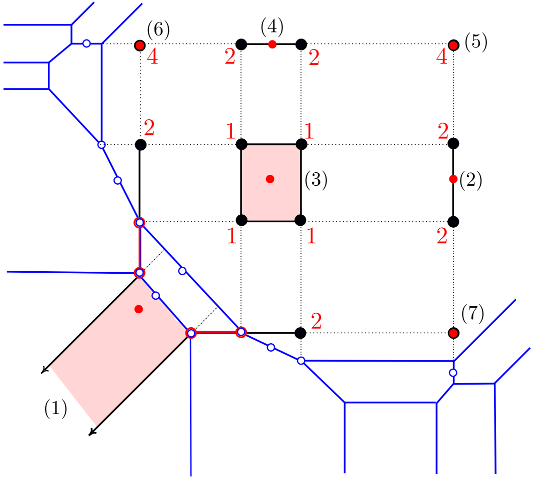
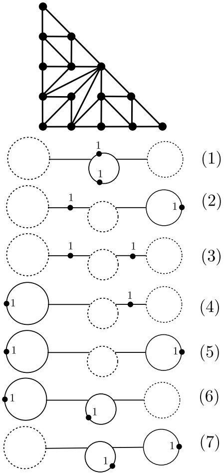




[BLMPR '16]: Equiv. class = move Λ continuously, remaining bitangent.
[L-M '18, J-M '20]: Each bitangent class lifts to 4 classical bitangents.

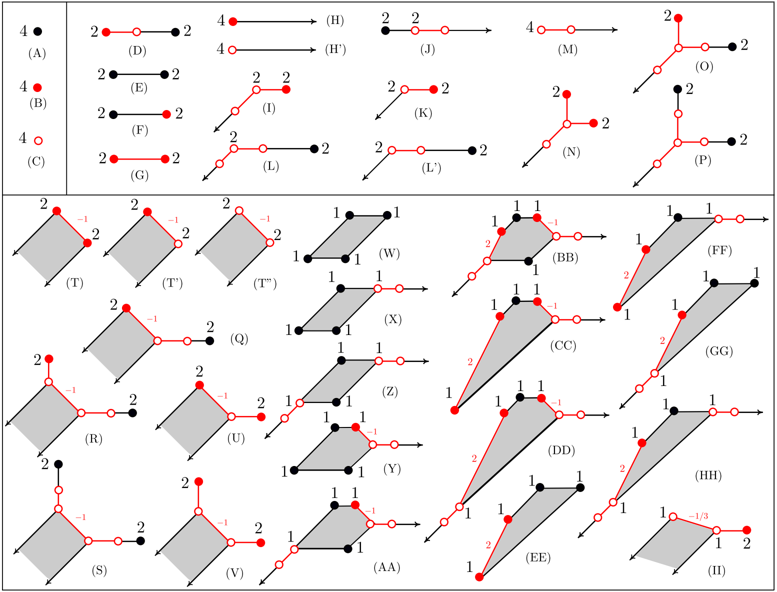
28 classical bitangents vs. 7 tropical bitangent classes.





C.-Markwig (2020): There are **39 shapes** of bitangent classes (up to symm.) They are **min-tropical** convex sets. Liftings come from vertices.
Over $\mathbb{R}$: liftings on each class are either all (totally) real or none is real.

THM 1: Classification into 39 bitangent classes (up to $\mathbb{S}_3$-symmetry)



Bitangent line ⊣ ⟷ location of its vertex (standard duality = -vertex)

Proof sketch of Combinatorial classification Theorem

Step 1: Identify edge directions for Γ involved in local tangencies.

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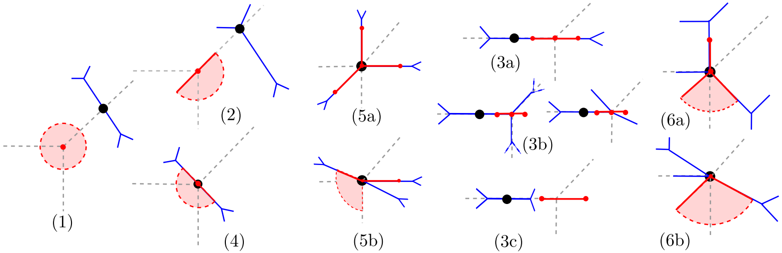

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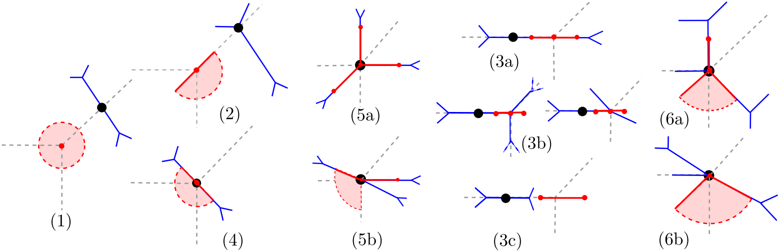
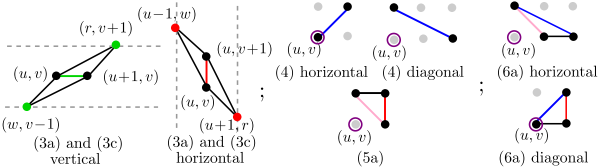


Step 3: Interpret $\mathbb{S}_3$-tangency types from cells in the Newton subdivision of $q(x,y) = \sum_{i,j} a_{i,j} x^i y^j$ with $\mathrm{Trop}(\mathcal{V}(q)) = \Gamma$ and combine local moves.



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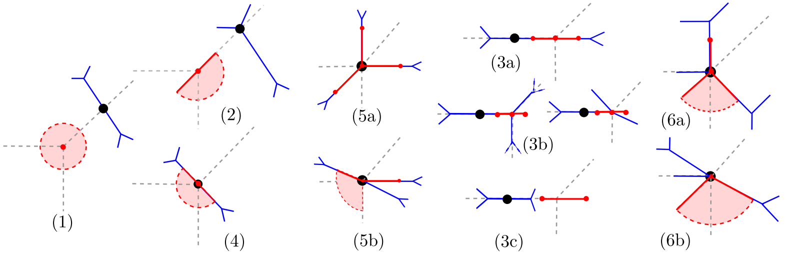

Step 3: Interpret $\mathbb{S}_3$-tangency types from cells in the Newton subdivision.

Step 4: Classify the shapes using 3 properties of its members:

max. mult.	proper	min. conn. comp.	shapes
4	yes	1	(II)
4	no	1	(C),(D),(L),(L'),(O),(P),(Q),(R),(S)
2	yes/no	2	rest

For the last row, refine using dimension and boundedness of its top cell.

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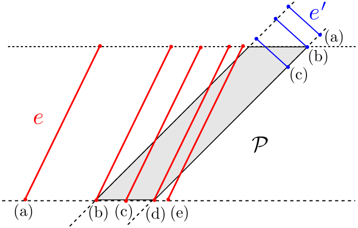

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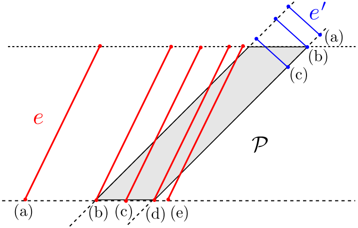
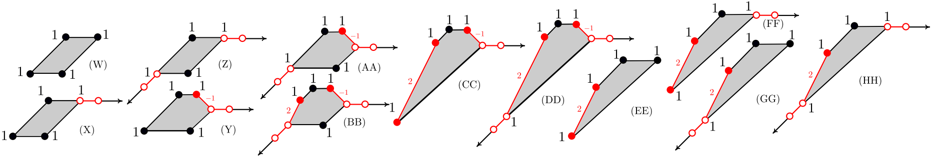
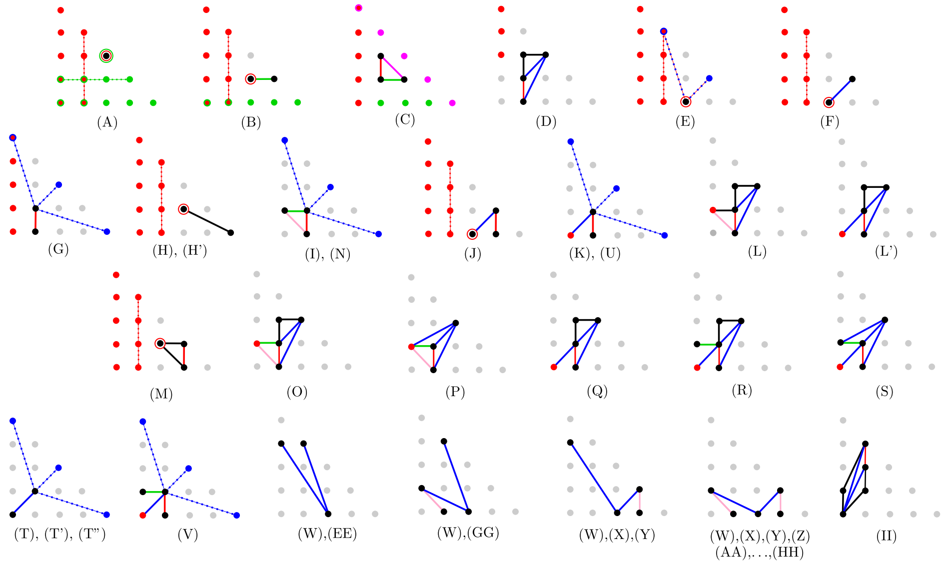


e' vs. e	(a)	(b)	(c)	(d)	(e)
(a)	(W)	(X)	(Y)	(GG)	(EE)
(b)	τ_1(X)	(Z)	(AA)	(HH)	(FF)
(c)	τ_1(Y)	τ_1(Z)	(BB)	(DD)	(CC)

$\tau_1 : X \mapsto -X,\ Y \mapsto Y - X$ in $\mathbb{R}^2$
($x \longleftrightarrow z,\ y \leftrightarrow y$ in $\mathbb{P}^2$)



Partial Newton subdivisions for all 39 bitangent shapes:



Lifting tropical bitangents to classical bitangents to $\mathcal{V}(q)$

Fix $\mathbb{K} = \mathbb{C}\{\{t\}\}$ (**complex** Puiseux series), $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$ (**real** P. s.)

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$$\boxed{\ell\colon\ y + m + n\,x = 0 \qquad \text{with } m, n \in \mathbb{K}^*.}$$

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Prop. [L-M '20]: If $p = (b_0\, t^{\alpha_0} + h.o.t, b_1\, t^{\alpha_1} + h.o.t)$, then

(i) $-(\alpha_0, \alpha_1)$ is a **trop. tangency pt.** for $\Lambda := \operatorname{Trop} \ell$ and $\Gamma := \operatorname{Trop} \mathcal{V}(q)$.

(ii) The initials $\bar{q}, \bar{\ell}, \bar{W}$ from **lowest valuation terms** of q, ℓ, W **vanish** at the initial term $\bar{p} := (b_0, b_1)$. (*Initial degener. vanish at $\bar{p}$!*)

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Thm. [L-M '20]: We can use $\bar{q} = \bar{\ell} = \bar{W} = 0$ to find $(\bar{m}, \bar{n}, \bar{p}) \in (\mathbb{C}^*)^4$.

Lifting tropical bitangents to classical bitangents (cont)

$$\boxed{(\bar{m}, \bar{n}, \bar{p}) \text{ and } \bar{q} = \bar{\ell} = \bar{W} = 0} \overset{???}{\rightsquigarrow} \boxed{(m, n, p) \text{ and } q = \ell = W = 0}$$

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Multivariate Hensel's Lemma: If $J_{x,y,\bar{m}}(\bar{q}, \bar{\ell}, \bar{W})_{|\bar{p}} \neq 0$, then $(\bar{m}, \bar{p})$ lifts to a **unique solution** (m, p); get n from $\ell(p) = 0$.

Lifting tropical bitangents to classical bitangents (cont)

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Crucial [C-M]: Lifting lies in $\mathbb{K}_{\mathbb{R}}$ if $(\bar{m}, \bar{n}, \bar{p}) \in \mathbb{R}^4$ and $q(x, y) \in \mathbb{K}_{\mathbb{R}}[x, y]$.

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type	(1)	(2)	(3a), (3b) or (3c)	(4)	(5a)	(6a)
mult.	0	1	2	$\lvert\det(e, e')\rvert$	2	$\lvert\det(e, e')\rvert$

(e' edge of Γ responsible for second tropical tangency, det $= 1$ or 2.)

Lifting tropical bitangents to classical bitangents (cont)

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[L-M'20, C-M'20]: If mult. four, no hyperflexes:

type	star	(5b)	(6b)
mult.	$2 \cdot 2$	1	1

Lifting tropical bitangents to classical bitangents (cont)

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[L-M'20, C-M'20]: If mult. four, no hyperflexes:

type	star	(5b)	(6b)
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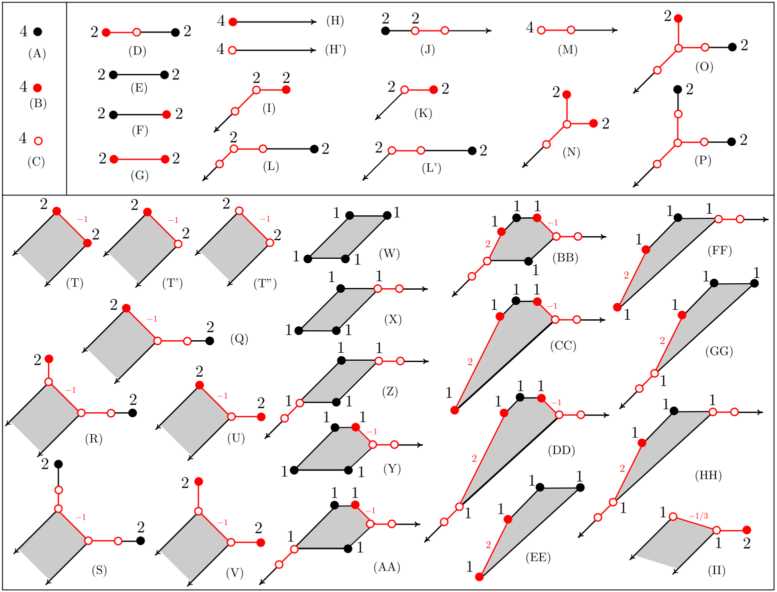
Thm.[L-M'20]: Local solns. for mult 1 in $\mathbb{Q}(\overline{a_{ij}})$ **but** for mult 2 in $\mathbb{Q}(\sqrt{\overline{a_{ij}}})$.

THM 2: Lifting multiplicities over $\mathbb{C}\{\{t\}\}$ for all 39 bitangent classes



THM 3: Total lifting multiplicity over $\mathbb{R}\{\{t\}\}$ for each shape is 0 or 4.

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Proof technique: determine when relevant radicands are positive and compare/combine constraints for different members of the same shape.

type	condition for real solutions	coeff.	end of Λ
(3a)	$(-1)^{w+v+1}(s_{uv}s_{u,v+1})^{w+v}s_{u-1,w}\, s_{u,v+1}\,\mathrm{sign}(\bar{n}) > 0$	m	horizontal
	$(-1)^{w+u+1}(s_{uv}s_{u+1,v})^{w+u}s_{w,v-1}\, s_{u+1,v}\,\mathrm{sign}(\bar{n}) > 0$	m/n	vertical
(3c)	$(-1)^{r+w}(s_{uv}s_{u,v+1})^{r+w}s_{u+1,r}\, s_{u-1,w} > 0$	m	horizontal
	$(-1)^{r+w}(s_{uv}s_{u+1,v})^{r+w}s_{r,v+1}\, s_{w,v-1} > 0$	m/n	vertical
(4),(6a)	$-\,\mathrm{sign}(\bar{n})s_{uv}s_{u+1,v+1} > 0$	m	diagonal
	$-\,\mathrm{sign}(\overline{m})s_{u,v+1}\, s_{u+2,v} > 0$	n	horizontal
(5a)	$\mathrm{sign}(\bar{n})s_{u+1,v}s_{u,v+1} > 0$	m	diagonal
	$\mathrm{sign}(\overline{m})s_{u+1,v+1}s_{u+1,v} > 0$	n	horizontal

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type	condition for real solutions	coeff.	end of Λ
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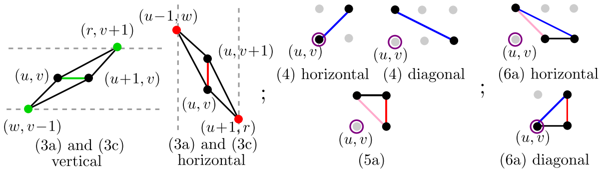
	$(-1)^{w+u+1}(s_{uv}s_{u+1,v})^{w+u}s_{w,v-1}\, s_{u+1,v}\operatorname{sign}(\bar{n}) > 0$	m/n	vertical
(3c)	$(-1)^{r+w}(s_{uv}s_{u,v+1})^{r+w}s_{u+1,r}\, s_{u-1,w} > 0$	m	horizontal
	$(-1)^{r+w}(s_{uv}s_{u+1,v})^{r+w}s_{r,v+1}\, s_{w,v-1} > 0$	m/n	vertical
(4),(6a)	$-\operatorname{sign}(\bar{n})s_{uv}s_{u+1,v+1} > 0$	m	diagonal
	$-\operatorname{sign}(\overline{m})s_{u,v+1}\, s_{u+2,v} > 0$	n	horizontal
(5a)	$\operatorname{sign}(\bar{n})s_{u+1,v}s_{u,v+1} > 0$	m	diagonal
	$\operatorname{sign}(\overline{m})s_{u+1,v+1}s_{u+1,v} > 0$	n	horizontal

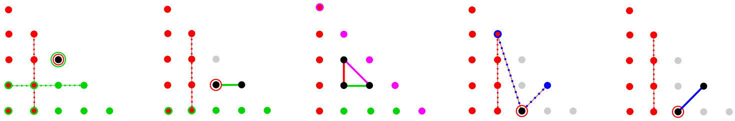
- s_{ij} = sign of initials $\overline{a_{ij}} \in \mathbb{R}$.
- Indices in formulas come from relevant cells in Newton subdivision:



Real lifting sign conditions for each representative bitangent class:

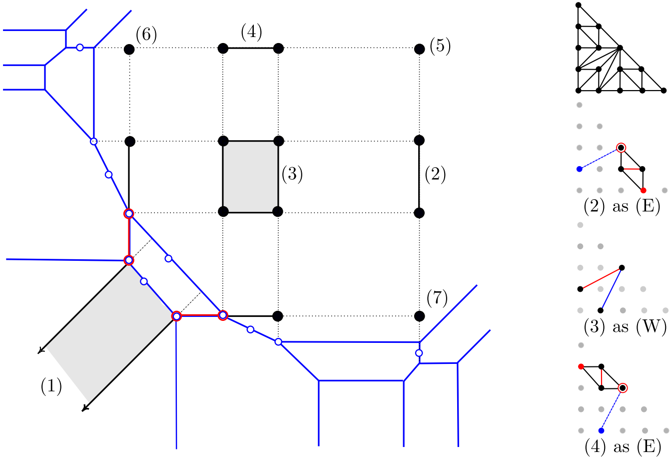
Shape	Lifting conditions
(A)	$(-s_{1v}s_{1,v+1})^{i}s_{0i}s_{22} > 0$ and $(-s_{u1}s_{u+1,1})^{j}s_{j0}s_{22} > 0$
(B)	$(-s_{1v}s_{1,v+1})^{i+1}s_{0i}s_{21} > 0$ and $(-s_{21})^{j+1}s_{31}{}^{j}s_{1v}s_{1,v+1}s_{j0} > 0$
(C)	$\begin{cases} (-s_{11}s_{12})^{i}s_{0i}s_{20} > 0 \text{ and } (-s_{21}s_{12})^{k}s_{k,4-k}s_{20} > 0 & \text{if } j = 2, \\ (-s_{11})^{i+1}s_{12}^{i}s_{21}s_{0i}s_{j0} > 0 \text{ and } (-s_{21})^{k+1}s_{12}^{k}s_{11}s_{k,4-k}s_{j0} > 0 & \text{if } j = 1, 3. \end{cases}$
(H),(H')	$(-s_{1v}s_{1,v+1})^{i+1}s_{0i}s_{21} > 0$ and $s_{1v}s_{1,v+1}s_{21}s_{40} < 0$
(M)	$(-s_{1v}s_{1,v+1})^{i+1}s_{0i}s_{21} > 0$ and $s_{1v}s_{1,v+1}s_{30}s_{31} > 0$
(D)	$(-s_{10}s_{11})^{i}s_{0i}s_{22} > 0$
(E),(F),(J)	$(-s_{1v}s_{1,v+1})^{i}s_{0i}\,s_{20} > 0$
(G)	$(-s_{10}s_{11})^{i}s_{0i}\,s_{k,4-k} > 0$
(I),(N)	$s_{10}s_{11}s_{01}s_{k,4-k} < 0$
(K),(T),(U),(V)	$s_{00}s_{k,4-k} > 0$
(L),(O),(P)	$s_{10}s_{11}s_{01}s_{22} < 0$
(L'),(Q),(R),(S)	$s_{00}s_{22} > 0$
rest	no conditions

Indices: relevant vertices in the Newton subdivision for each tangency, e.g.



(A) (B) (C) (E) (F)

Sample sign choices for our running example:



Negative signs	Real bitangent classes	Number of Real lifts	Topology
—	(1) and (3)	8	2 non-nested ovals
s_{31}	(1), (2), (3) and (7)	16	3 ovals
s_{13}, s_{31}	(1), . . . ,(7)	28	4 ovals
s_{13}, s_{31}, s_{22}	(3)	4	1 oval