Combinatorics and real lifts of bitangents to tropical plane quartics

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Joint work with Hannah Markwig (U. Tuebingen, Germany)


Algebraic Geometry Seminar UC Davis
Today’s focus: two classical result in Algebraic Geometry

Plücker (1834): A small quartic curve in $\mathbb{P}^2_{\mathbb{C}}$ has exactly 28 bitangent lines.

Zeuthen (1873): 4, 8, 16 or 28 real bitangents (real curve: $V_{\mathbb{R}}(f) \subset \mathbb{P}^2_{\mathbb{R}}$).

The real curve

- 4 ovals
- 1 oval
- 3 ovals
- 2 nested ovals
- 2 non-nested ovals
- empty curve

Trott: 28 totally real bitangents.

Salmon: 28 real, 24 totally real.

ISSUE: Plücker’s result fails tropically! But we can fix it.

GOAL: Use tropical geometry to find bitangents over $\mathbb{C}_\{t\}$ and $\mathbb{R}_\{t\}$.
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- What happens tropically?

Baker-Len-Morrison-Pflueger-Ren (2016): Every tropical smooth quartic in $\mathbb{R}^2$ has infinitely many tropical bitangents (in 7 equivalence classes.)

Conjecture [BLMPR]: Each bitangent class hides 4 classical bitangents.

- Two independent answers (with different approaches):
  - Len-Markwig (2020): We have an algorithm to reconstruct the 4 classical bitangents $\ell = y + mx + nx$ and the tangencies for each class under mild genericity conditions.

Question 1: What is a tropical bitangent line? Tropical tangencies?

Question 2: What is a tropical bitangent class?

Answer: Continuous translations preserving bitangency properties.
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Theorem: There are 28 classical bitangents to smooth plane quartics over $\mathbb{K} = \mathbb{C}((t))$ but 7 tropical bitangent classes to their smooth tropicalizations in $\mathbb{R}^2$. 

Tropical smooth quartic = dual to unimodular triangulation of $\Delta^2$ of side length 4. 

$\Rightarrow$ duality gives a genus 3 planar metric graph.
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Possible cases:
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**Possible cases:**

[BLOMP '16]

Basic facts about general tropical plane curves:

(1) Interpolation for general pts in $\mathbb{R}^2$ holds tropically (Mikhalkin's Corresp.) (unique line through 2 gen. points, unique conic through 5 gen. points, ...)

(2) General trop. curves intersect properly and as expected (Trop. Bézout.)
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Non-general case: Replace usual intersection with stable intersection.

$$C_1 \cap_{st} C_2 := \lim_{\varepsilon \to (0,0)} C_1 \cap (C_2 + \varepsilon).$$
Tropical bitangent Lines to tropical smooth quartics in $\mathbb{R}^2$:

**Definition:** \( \Lambda \) is a **bitangent line** for quartic \( \Gamma \) if and only if:

(i) \( \Lambda \cap \Gamma \) has 2 connected components of stable intersection multiplicity 2 each; or

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**[L-M ’20]**: 6 local tangency types between $\Lambda$ and $\Gamma$ (up to $S_3$-symmetry). 

Stable intersection (2 pts) vs. Midpoint tangency vs. Proper intersection

1. \( \begin{array}{c}
\bullet 2 \text{ or } 4 \end{array} \)

2. \( \begin{array}{c}
\bullet 2 \text{ or } 4 \end{array} \)

3a. \( \begin{array}{c}
2 \end{array} \)

3b. \( \begin{array}{c}
2 \text{ or } 4 \end{array} \)

3c. \( \begin{array}{c}
2 \end{array} \)

4. \( \begin{array}{c}
2 \text{ or } 4 \end{array} \)

5a. \( \begin{array}{c}
2 \end{array} \)

5b. \( \begin{array}{c}
4 \end{array} \)

6a. \( \begin{array}{c}
2 \end{array} \)

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- **Stable intersection (2 pts)**
- **Midpoint tangency**
- **Proper intersection**

\[\begin{align*}
(1) & \quad \begin{array}{c}
\text{2 or 4} \\
\end{array} \\
(2) & \quad \begin{array}{c}
\text{2 or 4} \\
\end{array} \\
(3a) & \quad 2 \\
(3b) & \quad 2 \text{ or 4} \\
(3c) & \quad 2 \\
(4) & \quad 2 \text{ or 4} \\
(5a) & \quad 2 \\
(5b) & \quad 4 \\
(6a) & \quad 2 \\
(6b) & \quad 4 \\
\end{align*}\]

- **Star shape**
28 classical bitangents vs. 7 tropical bitangent classes.
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Zharkov (2010): Trop. theta char on a metric graph $\Gamma \leftrightarrow H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$. 
$2\theta_i \sim K_\Gamma = \sum_{x \in \Gamma} (\text{val}(x) - 2)x$; $L_0$ non-effective $\leftrightarrow 0$; $2^{b_1(\Gamma)-1}$ effectives.
28 classical bitangents vs. 7 tropical bitangent classes.

[BLMPR '16]: 7 effective trop. theta characteristics on skeleton of tropical sm. quartic $\Gamma$ in $\mathbb{R}^2$ produce 7 tropical bitangent lines $\Lambda$ to $\Gamma$. 
28 classical bitangents vs. 7 tropical bitangent classes.

[BLMPR '16]: Equiv. class = move Λ continuously, remaining bitangent.

[L-M '18, J-M '20]: Each bitangent class lifts to 4 classical bitangents.
C.-Markwig (2020): There are 39 shapes of bitangent classes (up to symm.) They are min-tropical convex sets. Liftings come from vertices. Over \( \mathbb{R} \): lifttings on each class are either all (totally) real or none is real.
THM 1: Classification into 39 bitangent classes (up to $S_3$-symmetry)

Bitangent line $\leftrightarrow$ location of its vertex (standard duality $= -$vertex)
Step 1: Identify edge directions for $\Gamma$ involved in local tangencies.
Proof sketch of Combinatorial classification Theorem

**Step 1:** Identify edge directions for $\Gamma$ involved in local tangencies.

**Step 2:** Identify local moves of the vertex of $\Lambda$ that preserve one tangency.
Proof sketch of Combinatorial classification Theorem

**Step 1:** Identify edge directions for $\Gamma$ involved in local tangencies.

**Step 2:** Identify local moves of the vertex of $\Lambda$ that preserve one tangency.

**Step 3:** Interpret $S_3$-tangency types from cells in the Newton subdivision of $q(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ with $\text{Trop}(\mathcal{V}(q)) = \Gamma$ and combine local moves.
Proof sketch of Combinatorial classification Theorem

Step 1: Identify edge directions for $\Gamma$ involved in local tangencies.

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Step 3: Interpret $S_3$-tangency types from cells in the Newton subdivision.

Step 4: Classify the shapes using 3 properties of its members:

<table>
<thead>
<tr>
<th>max. mult.</th>
<th>proper</th>
<th>min. conn. comp.</th>
<th>shapes</th>
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<tbody>
<tr>
<td>4</td>
<td>yes</td>
<td>1</td>
<td>(II)</td>
</tr>
<tr>
<td>4</td>
<td>no</td>
<td>1</td>
<td>(C), (D), (L), (L'), (O), (P), (Q), (R), (S)</td>
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<tr>
<td>2</td>
<td>yes/no</td>
<td>2</td>
<td>rest</td>
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For the last row, refine using dimension and boundedness of its top cell.
Sample refinement: max mult. 2, dim=2 and bounded top-cell.
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• Since 2-cell is bounded, the tangency points for any member $\Lambda$ occur in relative interior of two different ends of $\Lambda$ (e.g. horizontal and diagonal).
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- \( \text{dim} \ 2 \) means we can find tangencies at two bounded edges \( e, e' \) of \( \Gamma \), both in the boundary of the conn. component of \( \mathbb{R}^2 \setminus \Gamma \) dual to \( x^2 \) (because \( e \) and \( e' \) are bridges of \( \Gamma \), so metric graph is \( \circ - \circ - \circ \)).
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- dim 2 means we can find tangencies at two bounded edges $e, e'$ of $\Gamma$, both in the boundary of the conn. component of $\mathbb{R}^2 \setminus \Gamma$ dual to $x^2$ (because $e$ and $e'$ are bridges of $\Gamma$, so metric graph is $\circ - \circ$).
- Draw parallelogram $\mathcal{P}$ with horizontal and diagonal lines through endpoints of $e$ and $e'$, respectively; analyze $\mathcal{P} \cap e$ and $\mathcal{P} \cap e'$
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<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
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<tr>
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<td>(X)</td>
<td>(Y)</td>
<td>(GG)</td>
<td>(EE)</td>
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<tr>
<td>(b)</td>
<td>$\tau_1(X)$</td>
<td>(Z)</td>
<td>(AA)</td>
<td>(HH)</td>
<td>(FF)</td>
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<td>(c)</td>
<td>$\tau_1(Y)$</td>
<td>$\tau_1(Z)$</td>
<td>(BB)</td>
<td>(DD)</td>
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$\tau_1: X \mapsto -X$, $Y \mapsto Y - X$ in $\mathbb{R}^2$

$(x \leftrightarrow z, y \leftrightarrow y$ in $\mathbb{P}^2)$
Partial Newton subdivisions for all 39 bitangent shapes:
Fix $K = \mathbb{C}\{t\}$ (complex Puiseux series), $K_R = \mathbb{R}\{t\}$ (real P. s.).
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- If $a = a_0 t^\alpha + h.o.t. \in K$, write $\bar{a} := a_0 = a t^{-\alpha}$ in $\mathbb{C}$ (initial term).
Lifting tropical bitangents to classical bitangents to $\mathcal{V}(q)$

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- If $a = a_0 t^\alpha + \text{h.o.t.} \in K$, write $\bar{a} := a_0 = \frac{a}{t^{-\alpha}}$ in $\mathbb{C}$ (initial term).

- Assume no classical bitangent line $\ell$ to $\mathcal{V}(q) \subset (K^*)^2$ is vertical and all tangency points are in torus (if not, rotate and translate). Thus, $\ell: y + m + n x = 0$ with $m, n \in K^*$. 

Question: When is $\ell$ tangent to $\mathcal{V}(q)$ at $p \in (K^*)^2$?

Answer: $p$ satisfies $\ell = q = W = 0$, where $W = J(\ell, q)$ is the Wronskian.
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- If $a = a_0 t^\alpha + h.o.t. \in K$, write $\bar{a} := a_0 = \overline{a t^{-\alpha}}$ in $\mathbb{C}$ (initial term).

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**Prop. [L-M '20]:** If $p = (b_0 t^{\alpha_0} + h.o.t, b_1 t^{\alpha_1} + h.o.t)$, then

(i) $-(\alpha_0, \alpha_1)$ is a trop. tangency pt. for $\Lambda := \text{Trop } \ell$ and $\Gamma := \text{Trop } \mathcal{V}(q)$.

(ii) The initials $\bar{q}, \bar{\ell}, \bar{W}$ from lowest valuation terms of $q, \ell, W$ vanish at the initial term $\bar{p} := (b_0, b_1)$. (Initial degener. vanish at $\bar{p}$!)
Fix $K = \mathbb{C}\{\{t\}\}$ (complex Puiseux series), $K_R = \mathbb{R}\{\{t\}\}$ (real Puiseux series).

- If $a = a_0 t^\alpha + h.o.t. \in K$, write $\bar{a} := a_0 = \overline{a t^{-\alpha}}$ in $\mathbb{C}$ (initial term).

- Assume no classical bitangent line $\ell$ to $V(q) \subset (K^*)^2$ is vertical and all tangency points are in torus (if not, rotate and translate). Thus, $\ell : y + m + n x = 0$ with $m, n \in K^*$.

**Question:** When is $\ell$ tangent to $V(q)$ at $p \in (K^*)^2$?

**Answer:** $p$ satisfies $\ell = q = W = 0$, where $W = J(\ell, q)$ is the Wronskian.

**Prop. [L-M '20]:** If $p = (b_0 t^{\alpha_0} + h.o.t, b_1 t^{\alpha_1} + h.o.t)$, then

(i) $-(\alpha_0, \alpha_1)$ is a trop. tangency pt. for $\Lambda := \text{Trop} \ell$ and $\Gamma := \text{Trop} V(q)$.

(ii) The initials $\bar{q}, \bar{\ell}, \bar{W}$ from lowest valuation terms of $q, \ell, W$ vanish at the initial term $\bar{p} := (b_0, b_1)$. ($\text{Initial degener. vanish at } \bar{p}!$)

**Thm. [L-M '20]:** We can use $\bar{q} = \bar{\ell} = \bar{W} = 0$ to find $(\bar{m}, \bar{n}, \bar{p}) \in (\mathbb{C}^*)^4$. 

M.A. Cueto (Ohio State)  Tropical Bitangents to Plane Quartics  May 6th 2020  22 / 27
Lifting tropical bitangents to classical bitangents (cont)

$$ (\bar{m}, \bar{n}, \bar{p}) \text{ and } \bar{q} = \bar{l} = \bar{W} = 0 \rightsquigarrow (m, n, p) \text{ and } q = \ell = W = 0 $$
Lifting tropical bitangents to classical bitangents (cont)

$$(\bar{m}, \bar{n}, \bar{p}) \text{ and } \bar{q} = \bar{\ell} = \bar{W} = 0 \quad \overset{??}{\longrightarrow} \quad (m, n, p) \text{ and } q = \ell = W = 0$$

**Multivariate Hensel’s Lemma:** If $J_{x,y}(\bar{q}, \bar{\ell}, \bar{W})|_{\bar{p}} \neq 0$, then $(\bar{m}, \bar{p})$ lifts to a unique solution $(m, p)$; get $n$ from $\ell(p) = 0$. 

Crucial [C-M]: Lifting lies in $K_R$ if $(\bar{m}, \bar{n}, \bar{p}) \in R^4$ and $q(x,y) \in K_R[x,y]$. 

[L-M '20]: Analyzed local mult. 2 tangencies and saw:

(i) Tangencies in 2 ends of $\Lambda$ give complementary data $(\bar{m}, \bar{n} \text{ or } \bar{m}/\bar{n})$.

(ii) Tangencies in same end of $\Lambda$ with $\Lambda \cap \Gamma$ disconnected give non-compatible local equations (genericity condition).

<table>
<thead>
<tr>
<th>Type</th>
<th>Mult.</th>
<th>$\det(e, e')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
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</tr>
<tr>
<td>(2)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(3a)</td>
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<td>1</td>
</tr>
<tr>
<td>(3b)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(3c)</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(4)</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(5a)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(5b)</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(6a)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(6b)</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

[L-M'20, C-M'20]: If mult. four, no hyperflexes:

- Type star

Thm. [L-M'20]: Local solns. for mult. 1 in $\mathbb{Q}(a_{ij})$ but for mult. 2 in $\mathbb{Q}(\sqrt{a_{ij}})$. 

M.A. Cueto (Ohio State) Tropical Bitangents to Plane Quartics May 6th 2020 23 / 27
Lifting tropical bitangents to classical bitangents (cont)

$$(\bar{m}, \bar{n}, \bar{p}) \quad \text{and} \quad \bar{q} = \bar{\ell} = \bar{W} = 0 \quad \Rightarrow \quad (m, n, p) \quad \text{and} \quad q = \ell = W = 0$$

**Multivariate Hensel’s Lemma:** If $J_{x,y,\bar{m}}(\bar{q}, \bar{\ell}, \bar{W})|_{\bar{p}} \neq 0$, then $(\bar{m}, \bar{p})$ lifts to a unique solution $(m, p)$; get $n$ from $\ell(p) = 0$.

**Crucial [C-M]:** Lifting lies in $\mathbb{K}_R$ if $(\bar{m}, \bar{n}, \bar{p}) \in \mathbb{R}^4$ and $q(x, y) \in \mathbb{K}_R[x,y]$. 

---

**Thm. [L-M’20]:** Local solns. for mult 1 in $\mathbb{Q}(a_{ij})$ but for mult 2 in $\mathbb{Q}(\sqrt{a_{ij}})$. 

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M.A. Cueto (Ohio State) 
Tropical Bitangents to Plane Quartics 
May 6th 2020 23 / 27
Lifting tropical bitangents to classical bitangents (cont)

\((\bar{m}, \bar{n}, \bar{p})\) and \(\bar{q} = \bar{\ell} = \bar{W} = 0\) \(\Rightarrow\) \((m, n, p)\) and \(q = \ell = W = 0\)

Multivariate Hensel’s Lemma: If \(J_{x,y,m}(\bar{q}, \bar{\ell}, \bar{W})|_{\bar{p}} \neq 0\), then \((\bar{m}, \bar{p})\) lifts to a unique solution \((m, p)\); get \(n\) from \(\ell(p) = 0\).

Crucial [C-M]: Lifting lies in \(K_R\) if \((\bar{m}, \bar{n}, \bar{p}) \in \mathbb{R}^4\) and \(q(x, y) \in K_R[x, y]\).

[L-M ’20]: Analyzed local mult. 2 tangencies and saw:

(i) Tangencies in 2 ends of \(\Lambda\) give complementary data \((\bar{m}, \bar{n}\) or \(\bar{m}/\bar{n})\).

(ii) Tangencies in same end of \(\Lambda\) with \(\Lambda \cap \Gamma\) disconnected give non-compatible local equations (genericity condition.)
Lifting tropical bitangents to classical bitangents (cont)

\[(\bar{m}, \bar{n}, \bar{p}) \text{ and } \bar{q} = \bar{\ell} = \bar{W} = 0 \quad \Rightarrow \quad (m, n, p) \text{ and } q = \ell = W = 0\]

**Multivariate Hensel’s Lemma:** If \( J_{x,y,\bar{m}}(\bar{q}, \bar{\ell}, \bar{W})|_{\bar{p}} \neq 0 \), then \((\bar{m}, \bar{p})\) lifts to a unique solution \((m, p)\); get \(n\) from \(\ell(p) = 0\).

**Crucial [C-M]:** Lifting lies in \(K_R\) if \((\bar{m}, \bar{n}, \bar{p}) \in \mathbb{R}^4\) and \(q(x, y) \in K_R[x, y]\).

**[L-M ’20]:** Analyzed local mult. 2 tangencies and saw:

(i) Tangencies in 2 ends of \(\Lambda\) give complementary data \((\bar{m}, \bar{n} \text{ or } \bar{m}/\bar{n})\).

(ii) Tangencies in same end of \(\Lambda\) with \(\Lambda \cap \Gamma\) disconnected give non-compatible local equations (genericity condition.)

<table>
<thead>
<tr>
<th>type</th>
<th>(1)</th>
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<th>(3a), (3b) or (3c)</th>
<th>(4)</th>
<th>(5a)</th>
<th>(6a)</th>
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</thead>
<tbody>
<tr>
<td>mult.</td>
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<td>2</td>
<td>(\det(e, e'))</td>
<td>2</td>
<td>(\det(e, e'))</td>
</tr>
</tbody>
</table>

\((e' \text{ edge of } \Gamma \text{ responsible for second tropical tangency, } \det = 1 \text{ or } 2\).\)
Lifting tropical bitangents to classical bitangents (cont)

$$(\bar{m}, \bar{n}, \bar{p}) \text{ and } \bar{q} = \bar{\ell} = \bar{W} = 0 \quad \rightarrow \quad (m, n, p) \text{ and } q = \ell = W = 0$$

Multivariate Hensel’s Lemma: If $J_{x,y,\bar{m}}(\bar{q}, \bar{\ell}, \bar{W})|_{\bar{p}} \neq 0$, then $(\bar{m}, \bar{p})$ lifts to a unique solution $(m, p)$; get $n$ from $\ell(p) = 0$.

Crucial [C-M]: Lifting lies in $\mathbb{K}_{\mathbb{R}}$ if $(\bar{m}, \bar{n}, \bar{p}) \in \mathbb{R}^4$ and $q(x, y) \in \mathbb{K}_{\mathbb{R}}[x, y]$.

[L-M ’20]: Analyzed local mult. 2 tangencies and saw:

(i) Tangencies in 2 ends of $\Lambda$ give complementary data ($\bar{m}$, $\bar{n}$ or $\bar{m}/\bar{n}$).

(ii) Tangencies in same end of $\Lambda$ with $\Lambda \cap \Gamma$ disconnected give non-compatible local equations (genericity condition.)

<table>
<thead>
<tr>
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<th>(1)</th>
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<th>(4)</th>
<th>(5a)</th>
<th>(6a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mult.</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>det$(e, e')$</td>
<td>2</td>
<td>det$(e, e')$</td>
</tr>
</tbody>
</table>

$$(e' \text{ edge of } \Gamma \text{ responsible for second tropical tangency, det } \neq 1 \text{ or } 2.)$$

[L-M’20, C-M’20]: If mult. four, no hyperflexes:

<table>
<thead>
<tr>
<th>type</th>
<th>star</th>
<th>(5b)</th>
<th>(6b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mult.</td>
<td>2 $\cdot$ 2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Lifting tropical bitangents to classical bitangents (cont)

\[(\tilde{m}, \tilde{n}, \tilde{p})\] and \[\tilde{q} = \tilde{\ell} = \tilde{W} = 0\] \[\xrightarrow{??}\] \[(m, n, p)\] and \[q = \ell = W = 0\]

Multivariate Hensel’s Lemma: If \(J_{x,y,\tilde{m}}(\tilde{q}, \tilde{\ell}, \tilde{W})|_{\tilde{p}} \neq 0\), then \((\tilde{m}, \tilde{p})\) lifts to a unique solution \((m, p)\); get \(n\) from \(\ell(p) = 0\).

Crucial [C-M]: Lifting lies in \(K_R\) if \((\tilde{m}, \tilde{n}, \tilde{p}) \in \mathbb{R}^4\) and \(q(x, y) \in K_R[x, y]\).

[L-M ’20]: Analyzed local mult. 2 tangencies and saw:

(i) Tangencies in 2 ends of \(\Lambda\) give complementary data \((\tilde{m}, \tilde{n}\) or \(\tilde{m}/\tilde{n}\)).

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<table>
<thead>
<tr>
<th>type</th>
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<th>(4)</th>
<th>(5a)</th>
<th>(6a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mult.</td>
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<td>2</td>
<td>det(e, e’)</td>
<td>2</td>
<td>det(e, e’)</td>
</tr>
</tbody>
</table>

(e’ edge of \(\Gamma\) responsible for second tropical tangency, det = 1 or 2.)

[L-M’20, C-M’20]: If mult. four, no hyperflexes:

<table>
<thead>
<tr>
<th>type</th>
<th>star</th>
<th>(5b)</th>
<th>(6b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mult.</td>
<td>2 \cdot 2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Thm.[L-M’20]: Local solns. for mult 1 in \(\mathbb{Q}(\sqrt{a_{ij}})\) but for mult 2 in \(\mathbb{Q}(\sqrt{a_{ij}})\).
THM 2: Lifting multiplicities over $\mathbb{C}\{t\}$ for all 39 bitangent classes
THM 3: Total lifting multiplicity over $\mathbb{R}\{t\}$ for each shape is 0 or 4.

Proof technique: determine when relevant radicands are positive and compare/combine constraints for different members of the same shape.

\[ (3a) \]
\[ \begin{align*}
&w + v + 1 \left( s^u v^s u^v + 1 \right)^w + v^s u^v - 1 \left( s^u v^s u^v - 1 \right)^w \\
&\text{sign}(\bar{n}) > 0 \\
&\text{horizontal}
\end{align*} \]

\[ (3c) \]
\[ \begin{align*}
&w + u + 1 \left( s^u v^s u^v + 1 \right)^w + u^s v^u - 1 \left( s^u v^s u^v - 1 \right)^w \\
&\text{sign}(\bar{n}) > 0 \\
&\text{vertical}
\end{align*} \]

\[ (4),(6a) \]
\[ \begin{align*}
&w + w \left( s^u v^s u^v + 1 \right)^r + w^s u^w - 1 \left( s^u v^s u^v - 1 \right)^r \\
&\text{sign}(\bar{n}) > 0 \\
&\text{diagonal}
\end{align*} \]

\[ (5a) \]
\[ \begin{align*}
&w + w \left( s^u v^s u^v + 1 \right)^r + w^s u^w - 1 \left( s^u v^s u^v - 1 \right)^r \\
&\text{sign}(\bar{n}) > 0 \\
&\text{vertical}
\end{align*} \]

$s_{ij} = \text{sign of initials $a_{ij}$} \in \mathbb{R}$. 

Indices in formulas come from relevant cells in Newton subdivision.
THM 3: Total lifting multiplicity over $\mathbb{R}\{\{t\}\}$ for each shape is 0 or 4.

**Proof technique:** determine when relevant radicands are positive and compare/combine constraints for different members of the same shape.

<table>
<thead>
<tr>
<th>type</th>
<th>condition for real solutions</th>
<th>coeff.</th>
<th>end of $\Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3a)</td>
<td>$(-1)^{w+v+1}(s_{uv}s_{u,v+1})^{w+v}s_{u-1,w}s_{u,v+1}\text{sign}(\bar{n}) &gt; 0$</td>
<td>$m$</td>
<td>horizontal</td>
</tr>
<tr>
<td></td>
<td>$(-1)^{w+u+1}(s_{uv}s_{u+1,v})^{w+u}s_{w,v-1}s_{u+1,v}\text{sign}(\bar{n}) &gt; 0$</td>
<td>$m/n$</td>
<td>vertical</td>
</tr>
<tr>
<td>(3c)</td>
<td>$(-1)^{r+w}(s_{uv}s_{u,v+1})^{r+w}s_{u+1,r}s_{u-1,w} &gt; 0$</td>
<td>$m$</td>
<td>horizontal</td>
</tr>
<tr>
<td></td>
<td>$(-1)^{r+w}(s_{uv}s_{u+1,v})^{r+w}s_{r,v+1}s_{w,v-1} &gt; 0$</td>
<td>$m/n$</td>
<td>vertical</td>
</tr>
<tr>
<td>(4),(6a)</td>
<td>$-\text{sign}(\bar{n})s_{uv}s_{u+1,v+1} &gt; 0$</td>
<td>$m$</td>
<td>diagonal</td>
</tr>
<tr>
<td></td>
<td>$-\text{sign}(\bar{m})s_{u,v+1}s_{u+2,v} &gt; 0$</td>
<td>$n$</td>
<td>horizontal</td>
</tr>
<tr>
<td>(5a)</td>
<td>$\text{sign}(\bar{n})s_{u+1,v}s_{u,v+1} &gt; 0$</td>
<td>$m$</td>
<td>diagonal</td>
</tr>
<tr>
<td></td>
<td>$\text{sign}(\bar{m})s_{u+1,v+1}s_{u+1,v} &gt; 0$</td>
<td>$n$</td>
<td>horizontal</td>
</tr>
</tbody>
</table>
THM 3: Total lifting multiplicity over $\mathbb{R}\{\{t\}\}$ for each shape is 0 or 4.

**Proof technique:** determine when relevant radicands are positive and compare/combine constraints for different members of the same shape.

<table>
<thead>
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<th>type</th>
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<th>end of $\Lambda$</th>
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</thead>
<tbody>
<tr>
<td>(3a)</td>
<td>$(-1)^{w+v+1}(s_{uv}s_{u,v+1})^{w+v}s_{u-1,w}s_{u,v+1}\text{sign}(\bar{n}) &gt; 0$</td>
<td>$m$</td>
<td>horizontal</td>
</tr>
<tr>
<td></td>
<td>$(-1)^{w+u+1}(s_{uv}s_{u+1,v})^{w+u}s_{w,v-1}s_{u+1,v}\text{sign}(\bar{n}) &gt; 0$</td>
<td>$m/n$</td>
<td>vertical</td>
</tr>
<tr>
<td>(3c)</td>
<td>$(-1)^{r+w}(s_{uv}s_{u,v+1})^{r+w}s_{u+1,r}s_{u-1,w} &gt; 0$</td>
<td>$m$</td>
<td>horizontal</td>
</tr>
<tr>
<td></td>
<td>$(-1)^{r+w}(s_{uv}s_{u+1,v})^{r+w}s_{r,v+1}s_{w,v-1} &gt; 0$</td>
<td>$m/n$</td>
<td>vertical</td>
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<tr>
<td>(4),(6a)</td>
<td>$-\text{sign}(\bar{n})s_{uv}s_{u+1,v+1} &gt; 0$</td>
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<tr>
<td></td>
<td>$-\text{sign}(\bar{m})s_{u,v+1}s_{u+2,v} &gt; 0$</td>
<td>$n$</td>
<td>horizontal</td>
</tr>
<tr>
<td>(5a)</td>
<td>$\text{sign}(\bar{n})s_{u+1,v}s_{u,v+1} &gt; 0$</td>
<td>$m$</td>
<td>diagonal</td>
</tr>
<tr>
<td></td>
<td>$\text{sign}(\bar{m})s_{u+1,v+1}s_{u+1,v} &gt; 0$</td>
<td>$n$</td>
<td>horizontal</td>
</tr>
</tbody>
</table>

- $s_{ij} = \text{sign of initials } a_{ij} \in \mathbb{R}$.
- Indices in formulas come from relevant cells in Newton subdivision:
Real lifting sign conditions for each representative bitangent class:

<table>
<thead>
<tr>
<th>Shape</th>
<th>Lifting conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>((-s_1 v_1 s_{1,v+1})^{i} s_0 i s_{22} &gt; 0) and ((-s_1 u_1 s_{u+1,1})^{j} s_{j0} s_{22} &gt; 0)</td>
</tr>
<tr>
<td>(B)</td>
<td>((-s_1 v_1 s_{1,v+1})^{i+1} s_0 i s_{21} &gt; 0) and ((-s_{21})^{i+1} s_{31}^{j} s_1 v s_{1,v+1} s_{j0} &gt; 0)</td>
</tr>
<tr>
<td>(C)</td>
<td>((-s_{11} s_{12})^{i} s_0 i s_{20} &gt; 0) and ((-s_{21} s_{12})^{k} s_{k,4-k} s_{20} &gt; 0) if (j = 2), ((-s_{11})^{i+1} s_{12}^{i} s_{21} s_0 i s_{j0} &gt; 0) and ((-s_{21})^{k+1} s_{12}^{k} s_{11} s_{k,4-k} s_{j0} &gt; 0) if (j = 1, 3).</td>
</tr>
<tr>
<td>(H),(H')</td>
<td>((-s_1 v_1 s_{1,v+1})^{i+1} s_0 i s_{21} &gt; 0) and (s_1 v s_{1,v+1} s_{21} s_{40} &lt; 0)</td>
</tr>
<tr>
<td>(M)</td>
<td>((-s_1 v_1 s_{1,v+1})^{i+1} s_0 i s_{21} &gt; 0) and (s_1 v s_{1,v+1} s_{30} s_{31} &gt; 0)</td>
</tr>
<tr>
<td>(D)</td>
<td>((-s_1 v s_{1,v+1})^{i} s_0 i s_{22} &gt; 0)</td>
</tr>
<tr>
<td>(E),(F),(J)</td>
<td>((-s_1 v s_{1,v+1})^{i} s_0 i s_{20} &gt; 0)</td>
</tr>
<tr>
<td>(G)</td>
<td>((-s_{10} s_{11})^{i} s_0 i s_{k,4-k} &gt; 0)</td>
</tr>
<tr>
<td>(I),(N)</td>
<td>(s_{10} s_{11} s_{01} s_{k,4-k} &lt; 0)</td>
</tr>
<tr>
<td>(K),(T),(U),(V)</td>
<td>(s_{k,4-k} &gt; 0)</td>
</tr>
<tr>
<td>(L),(O),(P)</td>
<td>(s_{10} s_{11} s_{01} s_{22} &lt; 0)</td>
</tr>
<tr>
<td>(L'),(Q),(R),(S)</td>
<td>(s_{00} s_{22} &gt; 0)</td>
</tr>
</tbody>
</table>

Indices: relevant vertices in the Newton subdivision for each tangency, e.g.
Sample sign choices for our running example:

<table>
<thead>
<tr>
<th>Negative signs</th>
<th>Real bitangent classes</th>
<th>Number of Real lifts</th>
<th>Topology</th>
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</thead>
<tbody>
<tr>
<td>—</td>
<td>(1) and (3)</td>
<td>8</td>
<td>2 non-nested ovals</td>
</tr>
<tr>
<td>$s_{31}$</td>
<td>(1), (2), (3) and (7)</td>
<td>16</td>
<td>3 ovals</td>
</tr>
<tr>
<td>$s_{13}, s_{31}$</td>
<td>(1), ..., (7)</td>
<td>28</td>
<td>4 ovals</td>
</tr>
<tr>
<td>$s_{13}, s_{31}, s_{22}$</td>
<td>(3)</td>
<td>4</td>
<td>1 oval</td>
</tr>
</tbody>
</table>

$\text{(2) as (E)}$

$\text{(3) as (W)}$

$\text{(4) as (E)}$