Steinberg–Whittaker Localization

and affine Harish–Chandra bimodules

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Preview - the theorem

Let G be a reductive group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$.

- Beilinson–Bernstein localization relates $\mathfrak{g}$-modules to D-modules on partial flag varieties G/P.
- Will explain an alternative picture, relating $\mathfrak{g}$-modules to D-modules on 'partial Whittaker flag varieties' $G/N, \psi$.
- One extreme - for regular central characters, recover usual story on full flag variety
- Other extreme - for maximally singular central characters, equates $\mathfrak{g}$-modules with a 'categorical Steinberg module' for G.

Preview - methods and applications

Two ingredients (also give new proof of usual localization) from categorical representation theory.

- Categories generated by B-invariant vectors
- Parabolic induction for groups and Hecke categories

Applications

- Useful substitute for central characters for affine Lie algebras
- Calculating character sheaves.

Next up:

1. Set up and reminders about Lie algebra representations and D-modules
2. Reminder about regular Beilinson–Bernstein localization
3. Reminder about singular localization
4. Statement and discussion of Steinberg–Whittaker localization

Setup - Groups

- Fix opposite Borel subgroups B, B^- of G, with unipotent radicals N, N^-.
- Consider the obtained maximal torus

$$T = B \cap B^- \subset G.$$

  and Weyl group $W = N_G(T)/T$.
- We have the 'big cell', i.e. open embedding

$$N \times T \times N^- \subset G$$

  and associated triangular decomposition

$$\mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^- = \mathfrak{g}.$$

Setup - Lie algebras

- Write $Z \subset U(\mathfrak{g})$ for the center of the universal enveloping algebra.
- Recall its spectrum

$$\operatorname{Spec} Z \simeq \mathfrak{t}^*//W$$

via assigning to $\lambda \in \mathfrak{t}^*$ the central character of the Verma module

$$M_\lambda := \operatorname{ind}_{\mathfrak{t}\oplus\mathfrak{n}}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

which depends only on the dot W orbit of λ.

- Recall λ is *regular* if has trivial stabilizer in W, otherwise *singular*.

Setup - Lie algebras

- Consider associated (derived) category of $\mathfrak{g}$-modules with generalized central character λ

$$\mathfrak{g}\text{-mod}_{\lambda}\,.$$

- Will briefly consider category with strict central character λ

$$\mathfrak{g}\text{-mod}_{\overline{\lambda}}$$

Remark

For ease of notation, will assume throughout that λ lies on the character lattice of T.

Setup - D-modules

- If X is a smooth algebraic variety, then a D-module on X is a quasi-coherent sheaf $\mathcal{E}$ equipped with a flat connection ∇.
- Denote the (derived) category of D-modules on X by $D(X)$.
- If G acts on X, then the global sections of $\mathcal{E}$ carry an action by global vector fields, and in particular $\mathfrak{g}$.
- This admits a left adjoint

$$\mathrm{Loc} : \mathfrak{g}\text{-mod} \rightleftarrows D(X) : \Gamma,$$

Review - regular localization

Theorem

(Beilinson–Bernstein, 1981) If λ is regular, then localization onto G/B yields an equivalence

$$\mathfrak{g}\text{-mod}_{\overline{\lambda}} \simeq D(G/B).$$

The functor, for $\lambda = 0$ is literally Loc, otherwise a variant. Similarly:

Theorem

(B.-B.) The above functor prolongs to an equivalence

$$\mathfrak{g}\text{-mod}_{\lambda} \simeq D(G/N)^{T-mon},$$

Note the latter is the full subcategory of $D(G/N)$ generated under pullback by $D(G/B)$.

Review - singular localization

Basic idea for general case - degenerating

$$\text{regular weights} \rightsquigarrow \text{singular weights}$$

corresponds to passing

$$\text{full flag variety} \rightsquigarrow \text{partial flag varieties}$$

Setup - invariant D-modules

Various types of 'invariant' D-modules

$$D(X)^G, \quad D(X)^{G-mon}, \quad D(X)^{G,w}.$$

$$G \times X \xrightarrow{act} X \quad G \times X \xrightarrow{p} X$$

- A G-equivariant D-module on X is an $\mathcal{E}$ equipped with identifications $act^!\mathcal{E} \simeq p^!\mathcal{E}$ in $D(G \times X)$ satisfying some natural compatibilities.
- There is a tautological forgetful functor $D(X)^G \to D(X)$.
- A D-module on X is G-monodromic if it lies the full subcategory generated by the essential image.
- A G-weakly equivariant D-module on X is as above, but with $act^!\mathcal{E} \simeq p^!\mathcal{E}$ only in $QCoh(G) \otimes D(X)$.

Setup - invariant D-modules

Example

If $X = pt$, then:

- $D(X)^G \simeq D(pt/G) \simeq H_*(G)$-mod
- $D(X)^{G-mon} \simeq D(pt) \simeq \text{Vect}$
- $D(X)^{G,w} \simeq \text{Rep}(G)$.

Review - singular localization

The stabilizer of λ in W canonically identifies with the Weyl group of a parabolic $P \supset B$, and consider its Levi factorization

$$P = LN_P.$$

Theorem

(Bezrukavnikov–Mirkovic–Rumynin 2006, Backelin–Kremnizer 2013) There is an equivalence

$$\mathfrak{g}\text{-mod}_\lambda \simeq D(G/N_P)^{L,w,\lambda},$$

where the latter are D-modules on G/N_P which are (i) weakly equivariant for the right action of L on which (ii) the center Z_L of $U(\mathfrak{l})$ acts with generalized character λ.

Review - singular localization

Some comments:

- Equivalences are again given by some version of *Loc*
- One extreme - for λ regular, e.g. 0, recovers Beilinson–Bernstein
- Other extreme - for λ fixed by W, e.g. $-\rho$, it says

$$\mathfrak{g}\text{-mod}_{-\rho} \simeq D(G)^{G,w,-\rho}.$$

But more generally,

$$\mathfrak{g}\text{-mod} \simeq D(G)^{G,w},$$

so the equivalence in this case is less geometrically contentful (weakly equivariant D-modules are not a full subcategory of D-modules).

Steinberg–Whittaker localization

- Let us turn to Steinberg–Whittaker localization, first for definiteness again in the extreme case of central character $-\rho$.
- Fix a generic additive character $\psi : N^- \to \mathbb{G}_a$. One can form, for any N^--variety X, the category of ψ-equivariant sheaves

$$D(X/N^-, \psi).$$

- I.e., now have character twist

$$act^! \mathcal{E} \simeq \psi^! \text{“}e^z\text{”} \overset{!}{\otimes} p^! \mathcal{E}.$$

$$\approx f(nx) = \psi(n) f(x)$$

Steinberg–Whittaker localization

As before, one can define global section and localization functors fitting into an adjunction

$$\mathrm{Loc} : \mathfrak{g}\text{-mod} \rightleftarrows D(G/N^-, \psi) : \Gamma.$$

Theorem

(Campbell-D.) Loc restricts to a fully faithful embedding

$$\mathrm{Loc} : \mathfrak{g}\text{-mod}_{-\rho} \hookrightarrow D(G/N^-, \psi).$$

Steinberg–Whittaker localization

Let us provide some comments about

$$\text{Loc} : \mathfrak{g}\text{-mod}_{-\rho} \hookrightarrow D(G/N^-, \psi).$$

- Here are two examples of the behavior of the functor:

$$M_{-\rho} \mapsto Fun(BN^-)\text{“}e^{\psi}\text{''} \approx Fun(B \times N^-)$$

$$\text{Loc}\, U_{-\rho} \mapsto \text{Coind}_{B,w}^{G,w} Fun(BN^-)\text{“}e^{\psi}\text{''} \approx Fun(G \times N^-).$$

  In particular, the latter is a compact generator for the essential image.
- We will see a different description of the essential image later, namely as an analog of the Steinberg module of $G(\mathbb{F}_q)$.

Steinberg–Whittaker localization

- To prepare for the case general λ, recall there is a canonical bijection

$$\mathrm{Hom}(N, \mathbb{G}_a)/T \simeq \{P \supset B\}.$$

- Under this, ψ generic corresponds to G, and ψ trivial corresponds to B.
- In particular, just as one assigned a P to λ, one may assign a ψ.

Steinberg–Whittaker localization

With this ψ, one can prove

Theorem

(Campbell-D.) There is a fully faithful embedding

$$\mathfrak{g}\text{-mod}_\lambda \hookrightarrow D(G/N, \psi).$$

- Again, the functor is a variant of Loc.
- In the other extreme case, of λ regular, i.e. $\psi = 0$, this is the former embedding

$$\mathfrak{g}\text{-mod}_\lambda \simeq D(G/N)^{T-mon} \hookrightarrow D(G/N).$$

- However, for singular λ, this differs from the former localization theorem.

Next up

Some discussion of ingredients in the argument, highlighting:

1. Checking equivalences of categories on highest weight vectors
2. Parabolic induction for categorical representations

Methods

- Analysis is facilitated by 'G-actions' on both sides of the theorems, which should be intertwined.
- (E.g., checking a map of vector spaces is an isomorphism may be made easier if the map underlies a map of modules).
- What symmetry is present in the current situation?

Methods

- Informally, given a $g \in G$, one can precompose a representation of $\mathfrak{g}$ by $\mathrm{Ad}_{g^{-1}}$ to get an automorphism

$$g : \mathfrak{g}\text{-mod}_{-\rho} \to \mathfrak{g}\text{-mod}_{-\rho} .$$

- Similarly, one can left translate by g to obtain an automorphism

$$g_* : D(G/N, \psi) \to D(G/N, \psi).$$

Methods

- In fact, have many more operators: namely, the latter is given by

$$D(G/N^-, \psi) \xrightarrow{\delta_g \boxtimes -} D(G) \otimes D(G/N^-, \psi) \xrightarrow{m_*} D(G/N^-, \psi).$$

- I.e., in fact the 'algebra' $D(G)$ of D-modules on G under convolution, acts on $D(G/N^-, \psi)$.
- Can think of $D(G)$ as an analog of the group algebra $\mathbb{C}[G(\mathbb{F}_q)]$.
- Similarly, previous action on Lie algebra modules extends to

$$D(G) \otimes \mathfrak{g}\text{-mod}_{-\rho} \to \mathfrak{g}\text{-mod}_{-\rho},$$

and Loc intertwines these actions.

Methods

- General formalism of $D(G)$ actions on categories due to Gaitsgory, building on ideas of Beilinson–Drinfeld, etc.
- Important work done by Beraldo, Raskin, Ben-Zvi–Nadler, etc.

Methods

- As always in representation theory, want to check properties of maps on highest weight vectors.
- Given an action of $D(B)$ on a category $\mathcal{C}$, can form B-invariant vectors (i.e., B-equivariant objects)

$$\mathcal{C}^B.$$

- Two relevant examples - if B acts on a variety X, then

$$D(X)^B \simeq D(B\backslash X),$$

and the action of $D(B)$ on $\mathfrak{g}$-mod yields

$$\mathfrak{g}\text{-mod}^B \simeq (\mathfrak{g}, B)\text{-modules} \approx \text{Category } \mathcal{O}.$$

Methods

- When does it suffice to check on highest weight vectors?

Theorem

(Folklore) For a $D(G)$-module $\mathcal{C}$, the canonical map

$$D(G/B) \underset{D(B\backslash G/B)}{\otimes} \mathcal{C}^B \to \mathcal{C}$$

is fully faithful.

- I.e., can canonically 'reconstruct' a piece of the entire module from its B-invariants.
- Key geometric fact - G/B is proper.

Methods

- In our case, we can reconstruct all the relevant Lie algebra representations.

Proposition

(Campbell-D.) The smallest full subcategory of $\mathfrak{g}\text{-mod}$ *preserved by* $D(G)$ *containing* M_λ *is*

$$\mathfrak{g}\text{-mod}_\lambda .$$

- I.e., $D(G)$-action gives an alternative to central central characters (useful in affine type).

Methods

- So, we only have to compare the 'highest weight lines'

$$\mathrm{Loc} : \mathfrak{g}\text{-mod}^{B}_{-\rho} \to D(B\backslash G/N^{-}, \psi).$$

- Both are equivalent to Vect, and one checks the map sends a generator to a generator, so we're done.
- I.e., $\mathfrak{g}$-mod$_{-\rho}$ is equivalent to the subcategory of $D(G/N^{-}, \psi)$ generated by its B-invariants.

Remark

In the complex representation theory of $G(\mathbb{F}_q)$, the analogous vector space of B-invariants

$$Fun(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/N^{-}(\mathbb{F}_q), \psi)$$

is one-dimensional, and generates an irreducible $G(\mathbb{F}_q)$ submodule of $Fun(G(\mathbb{F}_q)/N^{-}(\mathbb{F}_q), \psi)$, namely the *Steinberg representation*.

Methods

- Can equivalently characterize the essential image in terms of the action of $D(N^-, -\psi\backslash G/N^-, \psi)$ through an appropriate quotient, i.e. as
$$D(G/N^-, \psi)^{St},$$

Methods

This finished our discussion of the most singular case, i.e. $-\rho$. What about general λ? Two approaches:

1. The equivalence of $D(B\backslash G/B)$-modules

$$\mathfrak{g}\text{-mod}_\lambda^B \simeq D(B\backslash G/N, \psi)$$

   is known by Soergel bimodule techniques (Milicic–Soergel, Ginzburg, Webster, Bezrukavnikov–Yun, etc.), so tensor up with $D(G/B)$ and we're done.

2. Alternative - deduce the above equivalence, and usual localization at the same time, from a parabolic induction argument.

Methods

- Given $L \leftarrow P \rightarrow G$, can define (bi)adjoint parabolic induction and restriction functors

$$\text{pind} : D(L)\text{-mod} \xrightarrow{\text{Res}} D(P)\text{-mod} \xrightarrow{\text{Ind}} D(G)\text{-mod}\,.$$

$$\text{J} : D(G)\text{-mod} \xrightarrow{\text{Res}} D(P)\text{-mod} \xrightarrow{\text{inv}^{N_P}} D(L)\text{-mod}$$

- So, given a $D(L)$-module $\mathcal{C}$ and $D(G)$-module $\mathcal{D}$, to give an $D(G)$-equivariant functor

$$\text{pind}\, \mathcal{C} \rightarrow \mathcal{D}$$

is the same as giving a $D(L)$-equivariant functor

$$\mathcal{C} \rightarrow \mathcal{D}^{N_P}$$

Methods

Example

If L acts on a variety X, then

$$\operatorname{pind} D(X) \simeq D(G \overset{P}{\times} X).$$

In particular,

$$\operatorname{pind} D(L) \simeq D(G \overset{P}{\times} P/N_P) \simeq D(G/N_P),$$

who we recognize from the usual singular localization.

Methods

- For general λ, for P as before λ is a maximally singular weight for L. Thus, we have

$$D(L/N_L^-, \psi)^{St} \simeq \mathfrak{l}\text{-mod}_\lambda \simeq D(L)^{L,w,\lambda}$$

- Parabolically induce:

$$D(G/N^-, \psi)^{St} \simeq \text{pind}\,\mathfrak{l}\text{-mod}_\lambda \simeq D(G/N_P)^{L,w,\lambda}.$$

- So, localization is asking us to relate the central term with $\mathfrak{g}\text{-mod}_\lambda$.

Methods

- But parabolic induction of Lie algebra representations is a $D(L)$-equivariant functor

$$\mathfrak{l}\text{-mod} \xrightarrow{\mathrm{pind}_{\mathfrak{l}}^{\mathfrak{g}}} \mathfrak{g}\text{-mod}^{N_P},$$

- i.e. by adjunction yields a $D(G)$-equivariant map

$$\mathrm{pind}\, \mathfrak{l}\text{-mod} \to \mathfrak{g}\text{-mod}\,.$$

Theorem

(Campbell-D.) The above restricts to an equivalence

$$\mathrm{pind}\, \mathfrak{l}\text{-mod}_\lambda \simeq \mathfrak{g}\text{-mod}_\lambda$$

- Proof - we show generation by Borel invariants survives parabolic induction (and restriction), so suffices to analyze B-invariant vectors, which can be done directly.

Applications and variants.

- Can considering induction and restriction functors for Hecke categories, and show they intertwine with those for groups. E.g.,

$$\mathfrak{g}\text{-mod}_{\lambda}^{B} \simeq D(B\backslash G/B) \underset{D(B\backslash P/B)}{\otimes} \mathfrak{l}\text{-mod}_{\lambda}^{B},$$

  which is an important identity previously available at the level of Grothendieck groups (Soergel, Williamson, etc.).
- Given a $D(G)$-module $\mathcal{C}$ which is dualizable, can produce its character sheaf

$$\chi_{\mathcal{C}} \in D(G)^{G}.$$

  Can relate parabolic induction functors to those for character sheaves, and deduce descriptions of $\chi_{\mathfrak{g}\text{-mod}_{\lambda}}$ in terms of the Springer sheaf (parabolic inductions of sign representations).

Applications and variants

- Similar localization theorems for (noncritical) affine Lie algebras $\widehat{\mathfrak{g}}_\kappa$. In particular, can define

$$\bigoplus_{\lambda \in W_{aff} \backslash \mathfrak{t}^*} \widehat{\mathfrak{g}}_\kappa\text{-mod}_\lambda \subset \widehat{\mathfrak{g}}_\kappa\text{-mod}.$$

  in terms of $D(LG)$-submodules generated by Verma modules.
- Above immediately implies a linkage principle for positive energy modules on which the finite dimensional center acts locally finitely, i.e.

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^{K_1, Z-l.f.}$$

  previously suggested by Yakimov (2007).

Applications and variants

- Yields categories of Harish–Chandra bimodules

$$_{\mu}\mathrm{HCh}_{\lambda} \subset \widehat{\mathfrak{g}}_{\kappa} \oplus \widehat{\mathfrak{g}}_{-\kappa-Tate}\text{-mod}^{\widetilde{LG}_{-Tate}}$$

  whose existence and properties were predicted by I. Frenkel–Malikov (1998).

The end.

Thanks for listening!