Coulomb Branches and Plane Curve Singularities

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joint w/

- O. Kivinen] math
- T. Dimofte
- J. Hilburn
- A. Oblomkov
- L. Rozansky] physics

Motivation

Consider $\mathcal{C} := \{f(x,t)=0\} \subset \mathbb{C}^2$ a plane curve that is singular at $(0,0)$.
The intersection of $S^3_r \subseteq \mathbb{C}^2$, for $r \ll 1$, and $\mathcal{C}$ is a link $\mathcal{L} \subset S^3$.

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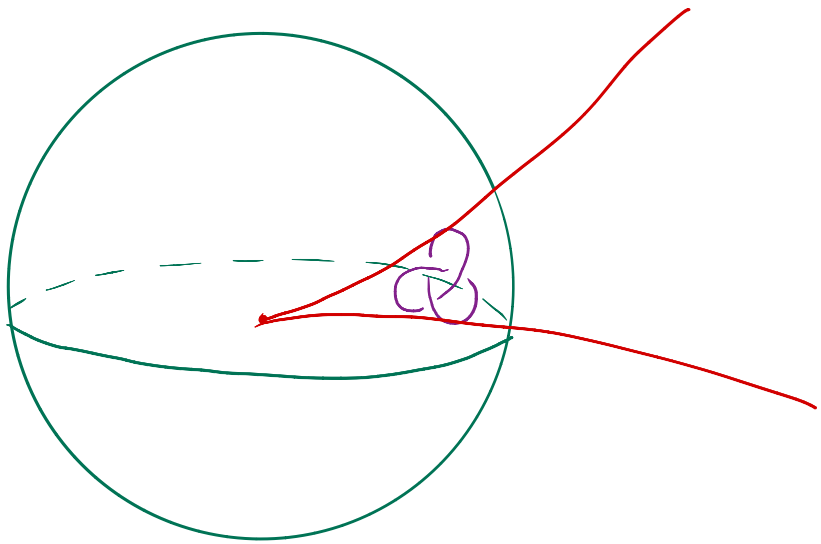

e.g. $f(x,t) = x^2 - t^3$

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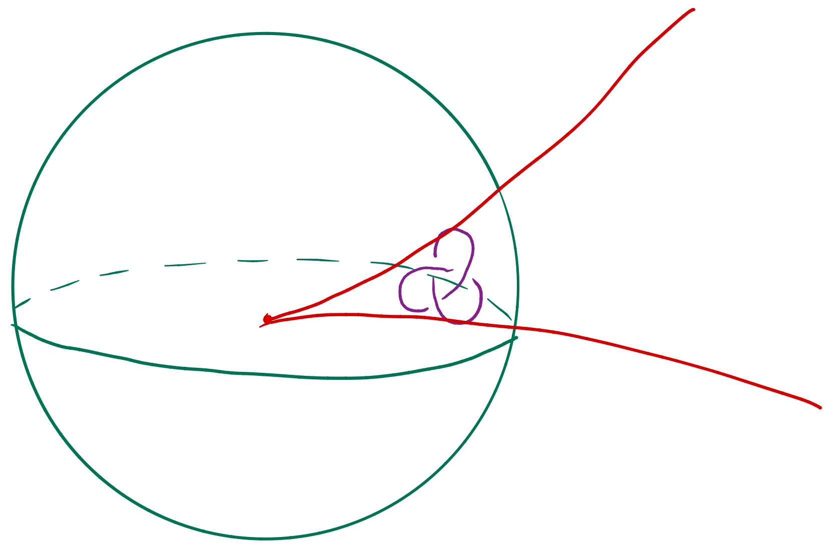

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Philosophy:

Geometry of $\mathcal{C}$ near $(0,0)$ encodes information about $\mathcal{L}$.

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Conjecture: (Oblomkov-Rasmussen-Shende)

For singularity with Milnor number $\mu \left(:= \dim_{\mathbb{C}} \mathbb{C}[[x,t]]/(\partial_x f, \partial_t f) \right)$

$$\overline{\mathcal{P}}_{min}(\mathcal{L}) = \left(\frac{1}{q}\right)^{\mu-1} \sum_{m \geq 0} q^{2m}\, w(\mathcal{C}^{[m]})$$

$w(X)$ = weight polynomial of X ~ graded euler character of $H^{\bullet}(X)$

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Note:

They also conjecture about the higher degree, these are determined by of higher moduli spaces $\mathcal{C}^{[m, m+l]}$.

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generators: $x \in \mathbb{C}[\mathfrak{h}]$, $\sigma \in \mathbb{C}[S^n]$, $y \in \mathbb{C}[\mathfrak{h}^*]$

relations:

i) $\sigma x \sigma^{-1} = \sigma(x)$

ii) $\sigma y \sigma^{-1} = \sigma(y)$

iii) $[x,y] = \langle y,x \rangle - c \sum_{\text{simple roots}} \langle y,\alpha \rangle \langle \alpha^\vee, x \rangle S_\alpha$

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Standard modules:

rep. of S_n

$$M_c(\tau) = \mathcal{H}_c \otimes_{\mathbb{C}[S_n] \otimes \mathbb{C}[\mathfrak{h}^*]} \tau \quad \text{w/ } y \cdot v = 0$$

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Theorem: (Berest-Etingof-Ginzburg)
$M_c(\tau)$ has a finite dimensional simple quotient iff $c = k/n$ for n, k coprime and either

i) $\tau = \text{trivial}$, $k > 0$ ii) $\tau = \text{sign}$, $k < 0$

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Prop: (Gorsky-Oblomkov-Rasmussen-Shende)

$$P_{\min}\left(\begin{matrix}(n,k)\text{-torus} \\ \text{knot}\end{matrix}\right) = \operatorname{tr}\left(q^h ; \operatorname{Hom}_{S_n}\left(\mathbb{C}, L_{k/n}\right)\right)$$

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Note:

There are higher a-degree versions of these statements given by replacing $\mathbb{C}$ by $\Lambda^{\ell}\mathbb{C}$.

Results (GK, in prep.)

S_n-invariant

Let $f(x,t)$ be of x-degree n, $e\overline{\mathcal{H}_c}e$ the spherical subalgebra of the $\mathfrak{gl}_n$ rational Cherednik algebra $\overline{\mathcal{H}_c}$.

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There is an action of $e\overline{\mathcal{H}}_c e$ on

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Remark:

There is a natural generalization to higher a-degree agreeing with the above conjectures.

Physics (DGHOR, in prep.)

- 3d $\mathcal{N}=4$ Gauge theory

$$G = GL(n,\mathbb{C}) \qquad N = \mathfrak{gl}(n,\mathbb{C}) \oplus \mathbb{C}^n \rightsquigarrow$$

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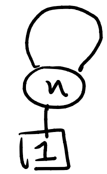

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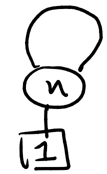
$$G = GL(n,\mathbb{C}) \qquad N = \mathfrak{gl}(n,\mathbb{C}) \oplus \mathbb{C}^n \rightsquigarrow$$ 

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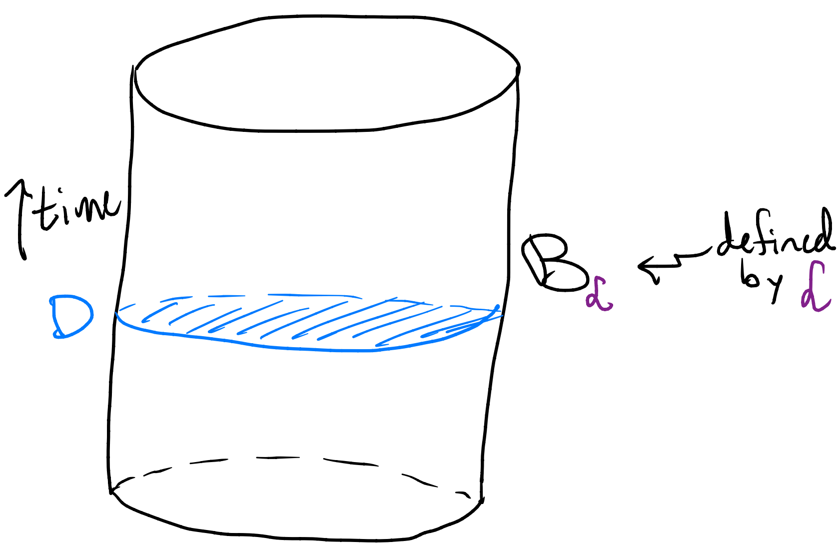

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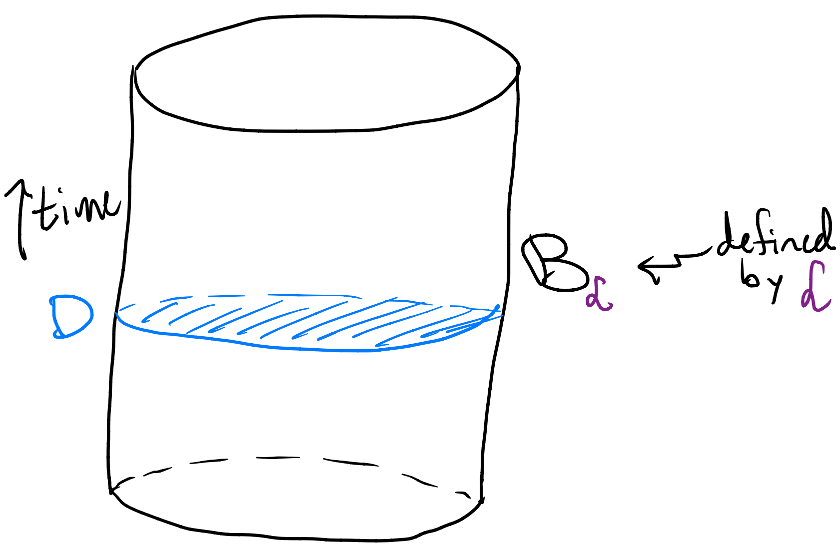

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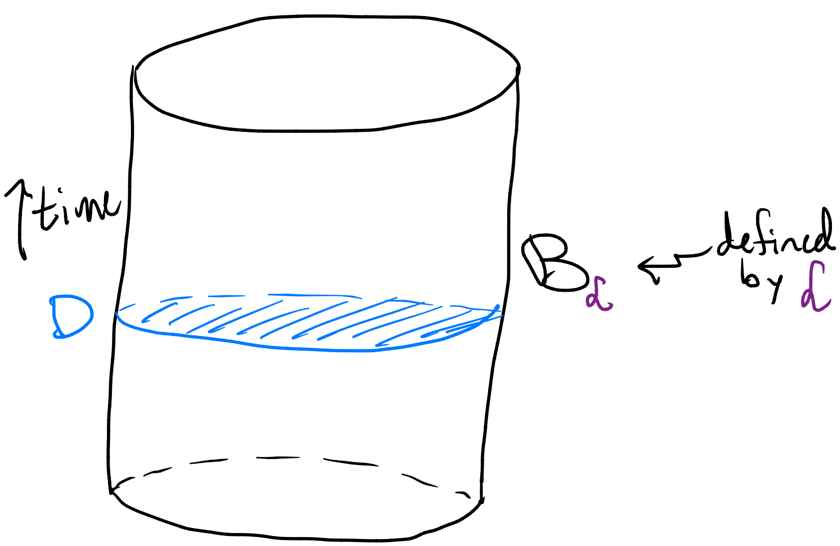

$$\tilde{A}_\hbar \circlearrowright \mathcal{H}_{susy}(D, B_\mathcal{L})$$

Conjecture:

$$\mathcal{H}_{susy}(D, B_\mathcal{L}) \cong H_{min}(\mathcal{L})$$

+ higher degree versions

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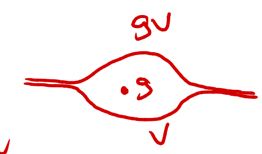
think: $\{$ 

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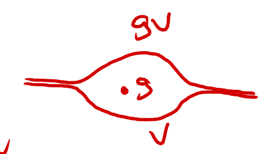
think: $\{$ 

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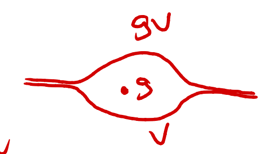
think: $\Big\{$ 

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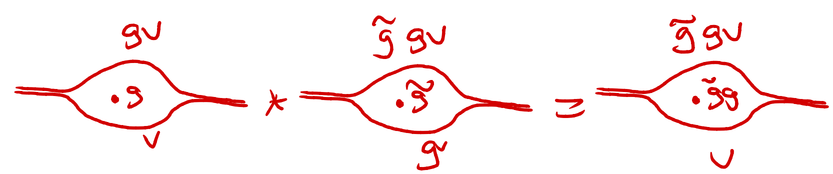
Think: 

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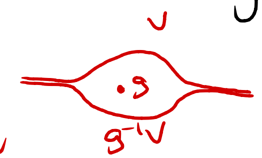
think: { 

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Think: 

$\mathcal{M}_C := \mathrm{Spec}\, \mathcal{A}$ "Coulomb Branch"

Note:

$\mathcal{A}$ has a natural quantization and admits a family of "flavor" deformations given by

$$1 \to G \to \tilde{G} \to T_F \to 1$$

for T_F a torus such that $\tilde{G} \circlearrowright N$.

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Ex:

- $G = \mathbb{C}^\times$, $N = \mathbb{C}^\ell$

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- $G = GL(n,\mathbb{C})$ $\quad N = \mathfrak{gl}(n,\mathbb{C}) \oplus \mathbb{C}^n$, $\tilde{G} = G \times \mathbb{C}^\times_{dil}$ scales $\mathfrak{gl}(n,\mathbb{C})$

$$\rightsquigarrow \tilde{A}_\hbar\big|_{\hbar=1} \cong e\,\overline{\mathcal{H}_m}\,e, \quad \tilde{\mathcal{M}}_c \cong \mathrm{Hilb}^n(\mathbb{C}^2)$$

BFN Springer Theory (Hilburn-Kamnitzer-Weekes)

Choose $v \in N_K$ and set $\mathcal{V}_v := (\tilde{G}_K^{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot} . v) \cap N_{\mathcal{O}}$, where $\tilde{G}_K^{\mathcal{O}}$ is the preimage of $(T_F)_{\mathcal{O}} \subset (T_F)_K$ in $\tilde{G}_K$.

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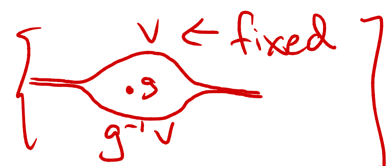
think: 

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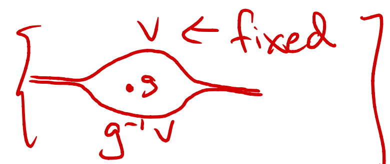
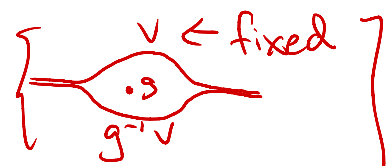


Theorem: (HKW)
There is an $\tilde{A}_\hbar$-module structure on $H_\bullet^{L_v}(Sp_v)$.

BFN Springer Theory (Hilburn - Kamnitzer - Weekes)

Choose $v \in N_K$ and set $V_v := (\tilde{G}_K^{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot} . v) \cap N_{\mathcal{O}}$, where $\tilde{G}_K^{\mathcal{O}}$ is the preimage of $(T_F)_{\mathcal{O}} \subset (T_F)_K$ in $\tilde{G}_K$.

Note:
If $\tilde{L}_v = \mathrm{stab}_{\tilde{G}_K^{\mathcal{O}} \rtimes \mathbb{C}^\times}(v)$ and $Sp_v := \{[g] \in Gr : g^{-1}v \in N_{\mathcal{O}}\}$

then $$V_v / G_{\mathcal{O}} = Sp_v / L_v$$ think: 

Theorem: (HKW)
There is an $\tilde{A}_\hbar$-module structure on $H_\bullet^{\tilde{L}_v}(Sp_v)$.

Note:
The deformation and quantization parameters may be forced to obey (linear) relations depending on $\tilde{L}_v$.

Back to Hilbert Schemes

Theorem: (GK, in prep)

Let $\mathcal{C} = \operatorname{Spec} \mathbb{C}[[x,t]]/(f(x,t))$ be the germ of a plane curve singularity. Without loss, assume tha

$$f(x,t) = x^n - f_{n-1}x^{n-1} - \cdots - f_0 .$$

(Weierstrauss preparation)

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Then for $G = GL(n,\mathbb{C})$, $N = \mathfrak{gl}_n \oplus \mathbb{C}^n$

$$C^{[\bullet]} := \bigsqcup_{m \geq 0} C^{[m]} \cong \mathrm{Spv}$$

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for $v = (e_1, \gamma)$, $\gamma = \begin{pmatrix} f_{n-1} & f_{n-2} & \cdots & f_1 & f_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$

Back to Hilbert Schemes

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Cor:

$H_*^{\tilde{L}_v}(\mathcal{C}^{[\bullet]})$ admits an action of $e\mathcal{H}_c e$ for some c.

Sketch of Proof:

First note that

c.f. Lusztig-Smelt

$$\mathcal{C}^{[\bullet]} = \left\{ \begin{array}{c} \text{nonzero} \\ \text{fractional } R \\ \text{ideals } I \end{array} \;\middle|\; I \subset R \right\} \cong \left\{ \begin{array}{c} \text{lattices } \Lambda \\ \text{closed under } \gamma \end{array} \;\middle|\; \Lambda \subset (\mathbb{C}^n)^* \right\} =: X$$

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Using $\Lambda \leftrightarrow [g]$ one finds $\Lambda = (\mathcal{O}^n)^* g^{-1} \in X$ if

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Similarly, if $g^{-1} \cdot (e_1, \gamma) \in \mathcal{N}_{\mathcal{O}}$ then $(\mathcal{O}^n)^* g^{-1} \in X$. Thus,

$$\mathcal{C}^{[\cdot]} \cong X \cong S_{pv}.$$

Remark:

For $f = x^n - t^k$, n,k coprime, we find $\tilde{L}_V \cong \mathbb{C}^\times$ has isolated fixed points labelled by monomial ideals. Moreover, it requires $c = -k/n$!

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Lemma: (GK, in prep.)

i) As a module for $e\overline{\mathcal{H}}_{-k/n}e$, $H^{\mathbb{C}^\times}_{\cdot}(\mathcal{e}^{[\cdot]}_{(n,k)})$ has a unique singular vector. $\left(\Rightarrow H^{\mathbb{C}^\times}_{\cdot}(\mathcal{e}^{[\cdot]}_{(n,k)}) \cong \mathbb{C}[x] \otimes M,\ M \in \operatorname{Rep} e\mathcal{H}_{k/n}e \text{ simple} \right)$

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ii) $\operatorname{grdim} H_\bullet^{\mathbb{C}^\times}(e^{[\cdot]}_{(n,k)}) = \frac{1}{1-q^n}\begin{bmatrix} n-1+k \\ n-1 \end{bmatrix}_q$ $\left(\Rightarrow \dim_{\mathbb{C}} M = \frac{1}{n}\binom{n+k-1}{n-1}\right)$

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Lemma: (GK, in prep.)

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Theorem/Cor: (GK, in prep.)

$$H_\bullet^{\mathbb{C}^\times}\left(e^{[\cdot]}_{(n,k)}\right) \cong e\overline{L_{-k/n}}$$

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For $f = x^n - t^k$, n,k coprime, we find $\tilde{T}_V \cong \mathbb{C}^\times$ has isolated fixed points labelled by monomial ideals. Moreover, it requires $c = -k/n$!

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Lemma: (GK, in prep.)

i) As a module for $e\overline{\mathcal{H}}_{-k/n}e$, $H_\cdot^{\mathbb{C}^\times}(e^{[\cdot]}_{(n,k)})$ has a unique singular vector. $\left(\Rightarrow H_\cdot^{\mathbb{C}^\times}(e^{[\cdot]}_{(n,k)}) \cong \mathbb{C}[x] \otimes M,\ M \in \mathrm{Rep}\, e\mathcal{H}_{k/n}e \text{ simple}\right)$

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Theorem/Cor: (GK, in prep.)

$$H_\cdot^{\mathbb{C}^\times}(e^{[\cdot]}_{(n,k)}) \cong e\overline{L}_{-k/n}$$

Proof:

Only one such module satisfying i), ii) exists.

Conclusions

① $\mathcal{C}^{[\cdot]}$ is a generalized affine Springer fiber (GASF) for $\mathcal{C}$ any plane curve singularity.

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Conclusions

① $\mathcal{C}^{[\cdot]}$ is a generalized affine Springer fiber (GASF) for $\mathcal{C}$ any plane curve singularity.

② Using a construction due to HKW, this gives an action of $e\mathcal{H}_c e$ on $H_\cdot^L(\mathcal{C}^{[\cdot]})$.

③ For $\mathcal{C} = \mathcal{C}_{(n,k)}$ corresponding to a torus knot, we have

$$H_\cdot^{\mathbb{C}^\times}(\mathcal{C}_{(n,k)}^{[\cdot]}) \cong e\overline{L_{-k/n}}.$$

Future Directions

1) Geometrically realize differential leading to KR homology

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3) Other colorings?

Thank
you!

BFN

Consider

$$([g_2, g_1 v], [g_1, v]) \longmapsfrom (g_2, [g_1, v]) \longmapsto [g_2, [g_1, v]] \longmapsto [g_2 g_1, v]$$

$$\begin{tikzcd}
\mathcal{T} \times \mathcal{R} & G_{\mathcal{K}} \times \mathcal{R} \arrow[l, "p"'] \arrow[r, "q"] & G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{R} \arrow[r, "m"] & \mathcal{T} \\
\mathcal{R} \times \mathcal{R} \arrow[u, hook] & p^{-1}(\mathcal{R} \times \mathcal{R}) \arrow[l] \arrow[r] \arrow[u, hook] & q(p^{-1}(\mathcal{R} \times \mathcal{R})) \arrow[r] \arrow[u, hook] & \mathcal{R} \arrow[u, hook]
\end{tikzcd}$$

Theorem: (Braverman, Finkelberg, Nakajima)

$c_1 \ast c_2 := (m \circ q)_* \circ p^*(c_1 \boxtimes c_2)$ gives $\mathcal{A} := (H^{G_{\mathcal{O}}}_{\bullet}(\mathcal{R}), \ast)$

the structure of a commutative algebra (over $\mathbb{C}$).

$\mathcal{M}_C := \operatorname{Spec} \mathcal{A}$ "Coulomb Branch"

HKW

Consider

$$\begin{tikzcd}
([g,v],v) & (g,v) \arrow[l, mapsto] \arrow[r, mapsto] & {[g,v]} \arrow[r, mapsto] & gv \\
T \times N_{\mathcal{O}} & G_K \times N_{\mathcal{O}} \arrow[l, "p"'] \arrow[r, "q"] & G_K \times_{G_{\mathcal{O}}} N_{\mathcal{O}} \arrow[r, "m"] & N_K \\
R \times N_{\mathcal{O}} \arrow[u, hook] & p^{-1}(R \times N_{\mathcal{O}}) \arrow[l] \arrow[r] \arrow[u, hook] & q(p^{-1}(R \times N_{\mathcal{O}})) \arrow[r] \arrow[u, hook] & N_{\mathcal{O}} \arrow[u, hook]
\end{tikzcd}$$

$\rightsquigarrow$ similar story to BFN

Note:

Using the map $\mu : T \to N_K$, $[g,v] \mapsto gv$ form $T \times_{N_K} T$,

then

$$R/G_{\mathcal{O}} \sim (T \times_{N_K} T)/G_K$$

$$[g,v] \sim ([g,v],[g',g'^{-1}gv]) \equiv ([g,v],[1,gv])$$

c.f. Steinberg variety

The convolution product on A should be thought of as coming from "$H^{G_K}(\mathcal{T} \times_{N_K} \mathcal{T})$." Moreover, for $v \in N_K$ expect, c.f. Springer theory,

$$H^{G_K}_\bullet(\mathcal{T} \times_{N_K} \mathcal{T}) \circlearrowright H^{\mathrm{stab}(v)}_\bullet(\mu^{-1}(v))$$

Theorem (HKW, in prep.)

$$A \circlearrowright H^{\mathrm{stab}(v)}_\bullet(\mu^{-1}(v)) \cong H_\bullet(\mathcal{M}_v)$$

$$\mathcal{M}_v := (G_K . v) \cap N_{\mathcal{O}} / G_{\mathcal{O}}$$