Coulomb Branches and Plane Curve Singularities

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UC Davis April 1st 2020


Motivation

Consider $e:=\{f(x, t)=0\} \subset \mathbb{C}^{2}$ a plane curve that is singular at $(0,0)$. The intersection of $S_{r}^{3} \leq \mathbb{C}^{2}$, for $r \ll 1$, and $C$ is a link $\mathcal{S} \subset \mathbb{S}^{3}$.

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e.g. $f(x, t)=x^{2}-t^{3}$
$\leadsto$ trefoil

Only depends on the type of singularity!

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Philosophy:
Geometry of $C$ near ( 0,0 ) encodes information about $\mathcal{L}$.

Denote $R=\mathbb{C}[[x, t]] / f(x, t)$ and $e^{\text {cans }}:=\{$ colength in ideals $I<R\}$

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Conjecture: (Oblomko-Rasmussen-Sheade)
For singularity with Minor number $\mu\left(:=\operatorname{dim}_{⿷} \mathbb{C}[(x,+]] /\left(a_{x} f, a_{\varepsilon} f\right)\right)$

$$
\bar{P}_{\min }(\mathcal{L})=\left(\frac{1}{q}\right)^{\mu-1} \sum_{m \geq 0} q^{2 m} w\left(e^{[m]}\right)
$$

$w(x)=$ weight polynomial of $X \sim$ graded of euler character
$\overline{P_{m i n}}:$ graded character of the lowest "a-degree"
(unreduced) triply graded HOMFLY homology

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Note:
They also conjecture about the higher degree, these are determined by of higher moduli spaces $e^{[m, m+e]}$.

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For torus knots $\left(f=x^{n}-t^{k}, \operatorname{gad}(n, k)=1\right)$ this can be taken further. Let $l_{c}$ be the sen spherical Cherednile algebra:
$\begin{array}{lccc}\text { generators: } & x & \sigma & y \\ & s & s & s \\ & \mathbb{C}[h] & \mathbb{C}\left[S^{n}\right] & \mathbb{C}\left[h^{*}\right]\end{array}$
relations: i) $\sigma \times \sigma^{-1}=\sigma(x)$
ii) $\sigma y \sigma^{-1}=\sigma(y)$
iii) $[x, y]=\langle y, x\rangle-c \sum_{\text {:ic ic }}\langle y, \alpha\rangle\left\langle\alpha^{v}, x\right\rangle S_{\alpha}$

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Standard modules:

$$
M_{c}(\tau)=M_{c}{\mathbb{C}\left(S_{n}\right] \otimes \mathbb{C}[k]}_{\otimes}^{t^{\text {rep. of } S_{n}}} \begin{aligned}
& \text { wy } y \cdot v=0
\end{aligned}
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Theorem: (Berest-Etingof-Ginzburg)
$\mu_{c}(\tau)$ has a finite dimensional simple quotient if $c=k / n$ for $n, k$ comprime and either
i) $\bar{\tau}=$ trivial, $k>0$
ii) $\tau=\operatorname{sign}, k<0$

Note:
The algebra $\mathcal{L}_{c}$ has an inner $\operatorname{se}(2,4)$ action with carton generator $h$.

Note:
The algebra $\mathcal{H}_{c}$ has an inner $\mathcal{E}(2, \mathbb{C})$ action with Carton generator $h$.

Prop: (Gorsky-Oblomkou-Rasmussen-Shende)
reduced, doubly-

$$
P_{\text {min }}\left(\begin{array}{c}
(n k k)-\text { tons } \\
k
\end{array} \text { of } n_{n}\right)=\operatorname{tr}\left(q^{h} ; \operatorname{Hom}_{s^{n}}\left(\mathbb{C}, L_{k / n}\right)\right)
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reduced, doublygraded HOMFLY polynomial
Conjecture: (Gorsky-Oblomkou-Rasmussen-Shende)
There exists a filtration $\mathcal{F}$ on $L_{m / n}$ such that

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H_{\min }\binom{(n, k) \text {-torus }}{k n o t} \cong \operatorname{Hom}_{S^{n}}\left(\mathbb{C}, g^{F} L_{k / n}\right) \begin{aligned}
& \text { reduced, triply- } \\
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Note:
There are higher $a$-degree versions of these statements given by replacing $\mathbb{C}$ by $\Lambda^{l} \mathbb{C}$.

Results (GK, in pere)
Let $f(x, t)$ be of $x$-degree $n, e{\overline{\rho_{c}} e \text { the spherical }}^{\text {Le }}$ subalgebre of the gen rational chereduik algebra $\mathrm{Ll}_{c}$.
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& e \overline{L l}_{c} e
\end{aligned} e \mu_{c} e \bigotimes_{\mathbb{C}}^{\otimes}(\mathbb{C}[x, y] /[x, y]=n)
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Theorem:
There is an action of $\overline{\mathrm{L}}_{c} e$ on

$$
H^{L} \cdot\left(e^{[0]}\right)=\underset{m \geqslant 0}{\oplus} H_{.}^{L}\left(e^{[m]}\right)
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In particular, for $f=x^{n}-t^{k} \quad(n, k)$ coprime

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H_{0}^{L}\left(e^{[\cdot]}\right) \cong e^{-k / n}
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Remark:
There is a natural generalization to higher a-degree agreeingwith the above conjectures.

Physics (DGHOR,in prep.)
$-3 \& \quad r=4$ Gange theory

$$
G=G L(n, \mathbb{C}) \quad N=g l(n, \mathbb{C}) \oplus \mathbb{C}^{n} \Leftrightarrow(n)
$$

Physics (DGHOR,in prep.)
-3d $r=4$ Gayye theory

$$
\begin{aligned}
G=G L(n, \mathbb{C}) \quad N & =g l(n, \mathbb{C}) \oplus \mathbb{C}^{n} \leadsto Q \\
\leadsto & M_{c} \cong M_{H}=H i l b^{n}\left(\mathbb{C}^{2}\right)
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$\widetilde{A}_{\hbar} \cong$ spherical subalgebra of the gln rational Cherednik algebra


$$
\tilde{A}_{\hbar} \because \mathcal{L}_{\text {susy }}\left(D, B_{2}\right)
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(Physics (DGHOR,in prep.)
-3d $\mathcal{K}=4$ Gauge theory

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G & =G L(n, \mathbb{C}) \quad N=g l(n, \mathbb{C}) \oplus \mathbb{C}^{n} \leadsto \\
\leadsto M_{c} \cong M_{H} & =H \cdot l b^{n}\left(\mathbb{C}^{2}\right)
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$\tilde{A}_{\hbar} \cong$ spherical subalgebra of the glen rational Cherednic algebra


$$
\tilde{A}_{\hbar} \subset \mathcal{L}_{\text {suss }}\left(D, B_{2}\right)
$$

Conjecture:

$$
\operatorname{l}_{\text {suss }}\left(D, B_{2}\right) \cong H_{\text {min }}(\alpha)
$$

thigher degree versions

Coulomb Branches
Choose $G$ complex reductive, $N \in R e p G$. Denote $O=\mathbb{C}[(t)], K=\mathbb{C}((t))$.
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Choose $G$ complex reductive, $N \in \operatorname{Rep} G$. Denote

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0=\mathbb{C}[(t)], K=\mathbb{C}((t)) \text {. }
$$

The space of triples $R$ is (as a set)

$$
Q=\left\{[g, T] \in G_{k} \times{ }_{G_{0}} N_{0} \mid v^{\prime}=g \cdot v \in N_{0}\right\} \subset G_{k} \times_{G_{0}} N_{0}=: \tau
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think: $\left\{\underset{\sim}{\sim} \frac{g v}{3 v} \quad v, g v \in N_{0}\right\}$
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& \text { think: }\{\underset{\sim}{\substack{3 v}} \text { v,gveNo }\}
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Theorem: (Braverman, Finkelberg, Nake jima) $A:=H^{a_{0}}(R)$ has the structure of a commutative algebra (over $\mathbb{C}$ ).

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& \left.R=\{[g,]] \in G_{k} \times{ }_{G_{0}} N_{0} \mid \quad g^{-1} v \in N_{v}\right\} \subset G_{K} \times_{G_{0}} N_{0}=: T \\
& \text { think: }\left\{\frac{g^{-N}}{\stackrel{v}{v}} v, g^{\prime \prime} v \in N_{0}\right\}
\end{aligned}
$$

Theorem: (Braverman, Finkelberg, Nake jima) $A_{0}:=H^{a_{0}}(R)$ has the structure of a commutative algebra (over $\mathbb{C}$ ). think:

$\mu_{c}:=\operatorname{Spec} A$ "Coulomb Branch"

Note:
A has a natural quantization and admits a family of "flavor" deformations given by

$$
1 \rightarrow G \rightarrow \widetilde{G} \rightarrow T_{F} \rightarrow 1
$$

for $T_{F}$ a torus such that $\widetilde{G} C N$.

$$
\widetilde{A}_{\hbar}:=H^{\tilde{G}_{0} \times C_{1}^{x}}(\Omega)
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Ex:

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\begin{aligned}
-G= & \mathbb{C}^{x}, N=\mathbb{C}^{l} \\
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-G & =G L(n, \mathbb{C}) \quad N=g e(n, \mathbb{c}) \oplus \mathbb{C}^{n}, \widetilde{G}=G \times \mathbb{C}_{\text {di }}^{x} \begin{array}{c}
\text { scales } \\
\text { gl e }(n, c)
\end{array} \\
& \left.\leadsto \tilde{A}_{\hbar}\right|_{\hbar=1} \cong e \overline{\mathcal{P}_{m}} e, \widetilde{\mu}_{c} \cong H_{i} \mid b^{n}\left(\mathbb{C}^{2}\right)
\end{aligned}
$$

(BFN Springer Theory (Hilbum-Kemnitzer)
Choose $v \in N_{K}$ and set $V_{v}:=\left(\tilde{G}_{K}^{0} \ngtr \mathbb{C}_{\text {rot. }}^{x}\right) \cap N_{O}$, where $\tilde{G}_{K}^{0}$ is the preimege of $\left(T_{F}\right)_{0} \subset\left(T_{F}\right)_{K}$ in $\tilde{G}_{K}$.
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Note:
If $\tau_{v}=\operatorname{stab} \tilde{G}_{q_{x}^{p} \times C^{x}}(v)$ and $\left.S_{p_{v}}:=\{g] \in G_{r}: g^{-0} \cup \in N O\right\}$
then $\quad v_{v} / G_{G}=S_{p v} / L_{v}$
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Theorem: (HKW)
There is an $\tilde{A}_{k}$-module structure on $H^{〔}$. (S $S_{\rho_{0}}$ ).
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Note:
If $\tau_{v}=\operatorname{sta}{\tilde{G}_{k} \times \mathbb{E}^{*}}(v)$ and $S_{p v}:=\left\{[g] \in G_{r}: g^{-1} v \in N_{0}\right\}$
then $\quad v_{v / G O}=S_{P_{v}} / L_{v}$

Theorem: (HKW)
There is an $\tilde{A}_{\hbar}$-module structure on $H^{\tilde{L}_{v}} \cdot\left(S_{P_{\nu}}\right)$.
Note:
The deformation and quantization parameters may be forced to obey (linear) relations depending on $\tilde{L}_{v}$.
(Back to Hilbert Schemes
Theorem: ( $G K$, in prep)
Let $e=\operatorname{Spec} \mathbb{C}[x, t]] /(f(x, t))$ be the germ of at plane curve singularity. Without loss, assume the

$$
f(x, t)=x^{n}-f_{n-1} x^{n-1}-\ldots-f_{0} .\binom{\text { Weierstranss }}{\text { preparation }}
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Then for $G=G L(n, 0), \quad N=g \ln \oplus \mathbb{C}^{n}$

$$
e^{[\cdot]}:=\bigsqcup_{m \geqslant 0} e^{[m]} \cong S_{p v}
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Then for $G=G L(n, 0), \quad N=g \ln \oplus \mathbb{C}^{n}$

$$
e^{[\cdot]}:=L_{m \geqslant 0} e^{[m]} \cong S_{p v}
$$

for $v=(e, \gamma), \quad \gamma=\left(\begin{array}{ccccc}f_{n-1} & f_{n-2} & \cdots & f_{1} & f_{0} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0\end{array}\right)$
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Cor:
$H_{*}^{\tilde{T v}_{v}}\left(e^{(\cdot \cdot)}\right)$ admits an action of $e \bar{H}_{c} e$ for some $c$.

Sketch of Proof:
First note that
c.f. Lusztig-Smelt

$$
e^{[\cdot]}=\{\underset{\substack{\text { nonzero } \\
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\text { c.f. Lusztig-Smelt } \\
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\end{array} \right\rvert\, \wedge c\left(O^{\wedge}\right)^{*}\right\}=: X
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Sketch of Proof:
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Using $\Lambda \leftrightarrow[g]$ one finds $\Lambda=\left(O^{n}\right)^{*} g^{-1} \in X$ if
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Similarly, if $g^{-1} \cdot\left(e_{1}, \gamma\right) \in \mathcal{N}_{0}$ then $\left(O^{n}\right)^{*} g^{-1} \in X$. Thus,

$$
e^{[\cdot]} \cong X \cong S_{p_{v}} .
$$

Remark:
For $f=x^{n}-t^{k}, n, k$ coprime, we find $\tau_{v} \cong \mathbb{C}^{x}$ has isolated fixed points labelled by monomial ideals. Moreover, it requires $c=-6 / n!$

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Lemma: (GK, in prep.)
i) As a module for $e \overline{\mathscr{M}_{-1 / n}} e, H^{\mathbb{C}^{x}} \cdot\left(e_{(n, k)}^{(\cdot \cdot)}\right)$ has a unique singular vector. $\left(\Rightarrow H^{\mathbb{C}^{x}}\left(e_{(1, x)}^{[\cdot]}\right) \cong \mathbb{C}[x] \otimes M, M \in \operatorname{Rep} e d x_{x / n}^{e}\right.$ simple)

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In the fixed-point basis we can show:
Lemma: (GK, in prep.)
i) As a module for $e \overline{\mathscr{M}_{-1 / n}} e, H^{\Phi^{x}} \cdot\left(e_{(n, k)}^{(\cdot-3)}\right)$ has a unique singular vector. $\left(\Rightarrow H^{\mathbb{C}^{x}}\left(e_{(n, k)}^{[\cdot]}\right) \cong \mathbb{C}[x] \otimes M, M \in \operatorname{Rep} e d x x_{x / n}^{e}\right.$
ii) $\operatorname{grdim} H^{C^{x}} .\left(e_{(n, 2)}^{(\sqrt{1})}\right)=\frac{1}{1-q^{n}}\left[\begin{array}{c}n-1+1 \\ n-1\end{array}\right]_{q}\left(\Rightarrow \operatorname{dim} \mathbb{C} M=\frac{1}{n}\binom{n+k-1}{n-1}\right)$

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In the fixed-point basis we can show:
Lemma: (GK, in prep.)
i) As a module for $e \overline{X_{-i / n}} e, H^{c^{x}} \cdot\left(e_{(0,0)}^{c_{3}}\right)$ has a unique singular vector. $\left(\Rightarrow H^{\mathbb{C}^{x}}\left(e_{(, x, x)}^{[3]}\right) \cong \mathbb{C}[x] \otimes M, M \in \operatorname{Rep} e+l_{x=n}^{e}\right.$
ii) $\operatorname{grdim} H^{\alpha^{C}} \cdot\left(e_{(n, 2)}^{(n)}\right)=\frac{1}{1-G^{(2}}\left[\begin{array}{c}n-1+1 \\ n-1\end{array}\right]_{q} \quad\left(\Rightarrow \operatorname{dim} c M=\frac{1}{n}\binom{n+k-1}{n-1}\right)$

Theorem/Cor: (GK, in prep.)

$$
H^{c^{k}} \cdot\left(e_{(0, k)}^{(\mathrm{cJ}}\right) \cong \bar{e}_{-v / n}
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Remark:
For $f=x^{n}-t^{k}, n, k$ coprime, we find $\tau_{v} \cong \mathbb{C}^{x}$ has isolated fixed points labelled by monomial ideals. Moreover, it requires $c=-6 / n$ !
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Lemma: (GK, in prep.)
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Theorem/Cor: (GK, in prep.)

$$
H^{\mathbb{C}^{k}} \cdot\left(e_{(n, k)}^{(\cdot 3)}\right) \cong \bar{e}_{-2 / n}
$$

Proof:
Only one such module satisfying i), ii) exists.

Conclusions
(1) $e^{[\cdot]}$ is a generalized affine $S$ pringer fiber (GASF) for $e$ any plane curve singularity.

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Conclusions
(1) $e^{[\cdot]}$ is a generalized affine Springer fiber (GASF) for $e$ any plane curve singularity.
(2) Using a construction due to HKW, this gives an action of $e \overline{X l}_{c} e$ on $H_{0}^{L}\left(e^{t^{[7}}\right)$.
(3) For $e=e_{(n, k)}$ corresponding to a torus knot, we have

$$
H^{d^{x}},\left(e_{(n, k)}^{[\cdot]}\right) \cong \bar{e} \overline{L-k / n} .
$$

Future Directions

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3) Other colorings?

Thank you!
$B F N$
Consider

$$
\begin{aligned}
& \left(\left[g_{2,}, g_{n}\right],\left[g_{1, v}\right]\right) \longleftrightarrow\left(g_{2},\left[g_{1}, v\right]\right) \mapsto\left[g_{2},\left[g_{1}, v\right]\right] \rightarrow\left[g_{2} g_{1}, v\right]
\end{aligned}
$$

$$
\begin{aligned}
& R \times R \longleftarrow P^{-1}(R \times R) \rightarrow q\left(p^{-1}(R \times R)\right) \rightarrow R
\end{aligned}
$$

Theorem: (Braverman, Finkelbery, Nakejima)

$$
C_{1} \not C_{2}:=(m \circ q)_{*} \circ p^{*}\left(c_{1} \otimes C_{2}\right) \text { gives } A^{0}:=\left(H^{a_{0}} \cdot(R), \otimes\right)
$$

the structure of a commutative algebra (over $\mathbb{C}$ ). $M_{c}:=\operatorname{Spec} A$ "Coulomb Branch"

HEW
Consider

$$
\begin{aligned}
& ([g, v], v) \longleftrightarrow(g, v) \mapsto g, v] \rightarrow g v \\
& \underset{\uparrow}{\tau \times N_{0}} \stackrel{p}{\stackrel{G}{k} \times N_{0}} \xrightarrow{q} \underset{\mathcal{q}_{k} \times N_{0}}{G_{0}} \xrightarrow{m} N_{k} \\
& R \times N_{0} \longleftarrow p^{-1}\left(R \times N_{0}\right) \rightarrow q\left(P^{-1}\left(R \times N_{0}\right)\right) \rightarrow N_{0}
\end{aligned}
$$

$\rightarrow$ similes story to BFN
Note:
Using the map $\mu: \tau \rightarrow N_{k},[g, \nu] \mapsto g V$ form $T \underset{N_{k}}{\times} \tau$, then

$$
\begin{aligned}
& R / G_{0} \sim(\tau \underset{v k}{\times} \tau) / G_{k} \quad \text { c.f. Steinberg } \\
& {[g, v] } \sim([g, v],[g, G T \cdot[J]) \equiv(g, v],[1, g, v) \\
& \text { variety }
\end{aligned}
$$

The convolution product on A should be thought of as coming from " $H^{G / k}\left(\tau x_{N k} \tau\right)$." Moreover, for $v \in N_{k}$ expect, c.f. Springer theory,

$$
H^{G k} \cdot\left(\tau \underset{N_{k}}{G^{2}} \tau\right) \quad \therefore H_{0}^{\operatorname{stab}(v)}\left(\mu^{-1}(v)\right)
$$

Theorem (HKW, in prep.)

$$
\begin{gathered}
A \subset H^{\operatorname{stan}(v)} \cdot\left(\mu^{-1}(v)\right) \cong H_{\cdot}\left(M_{v}\right) \\
M_{v}:=\left(G_{k} \cdot v\right) \cap N_{0} / G_{0}
\end{gathered}
$$

