WisCon - joint project Caprau - Sazdanovic
Lee - Zhang.
- Lowrance

Goal: **Thm:** let C be a ribbon concordance from $L_0 \to L_1$ then the induced sl_n-homology maps $H_n(C): H_n(L_0) \to H_n(L_1)$ are injective $n \geq 2$.

- Zemke: Knot-Floer-homology. 2019
- Levine-Zemke: Kh. 2019
- CGLLSZ : universal sl_2, sl_3; sl_n } 2020.
- Kang : KR, Khovanov like theories ... }

① sl_n-homology:

link diagram ⇝ bigraded homology theory.
link diagram ⇝ chain complex ⇝ bigraded homology theory.

$H_n(L) = \bigoplus H_n^{i,j}(L)$ } link invariant

} Euler charac.

sl_n-link polynomial. $\longleftrightarrow$ $\langle L \rangle = \sum (-1)^i q^j \operatorname{rank}(H_n^{i,j}(L))$

idea:
- each component of L ⇝ color V
  ↑ irrep of sl_2
- homology of the unknot → Frobenius alg A. of dim V

Foams
Bar-Natan
Khovanov
Stošić-Vaz

$\mathfrak{sl}_2 \longrightarrow \mathbb{Z}[X,a,b]/\underline{X^2 - aX - b}$

$\mathfrak{sl}_3 \longrightarrow \mathbb{Z}[X,a,b,c]/\underline{X^3 - aX^2 - bX - c}$

universal theories

Stošić-Vaz-S Foam theory

$\mathfrak{sl}_n \longrightarrow \mathbb{Z}[X]/X^n$

- a map between $S: L_0 \longrightarrow L_1$ induce a hom.

$$H(S): H(L_0) \longrightarrow H(L_1) \quad \text{up to signs.}$$

- the theory is functorial. (invariant under isotopy)

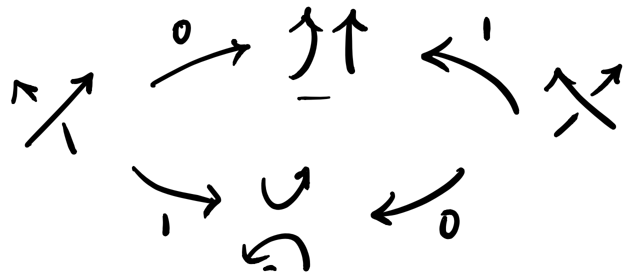
$$H(L_1 \sqcup L_2) = H(L_1) \otimes H(L_2)$$

cat of link cobordism $\longrightarrow$ cat bigraded abelian grps.

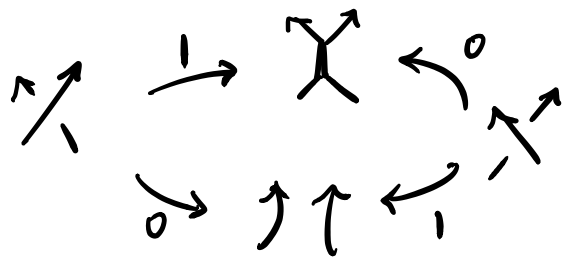
Game:

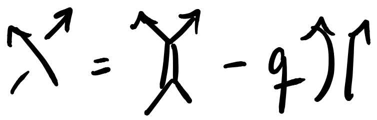
- Flatten diagram.

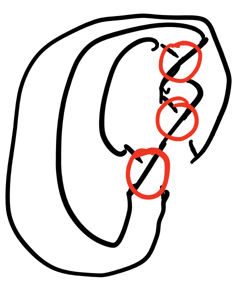
$\mathfrak{sl}_2$:



$(\mathfrak{sl}_n)$ $\mathfrak{sl}_3$:





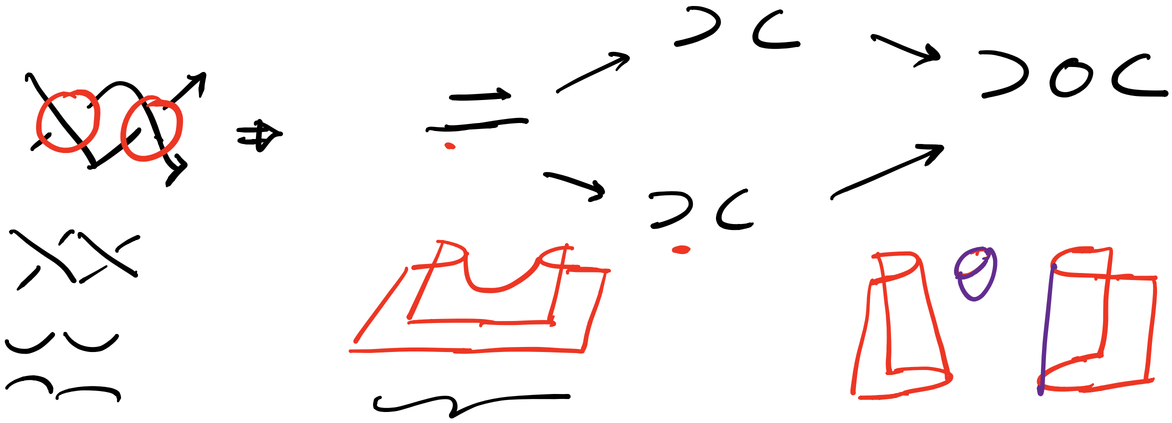
$L =$ 

$\longrightarrow 2^n \longleftarrow$ # crossings of the diagram for L.

pictures

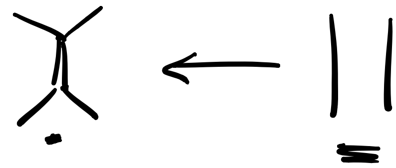
cube of resolutions

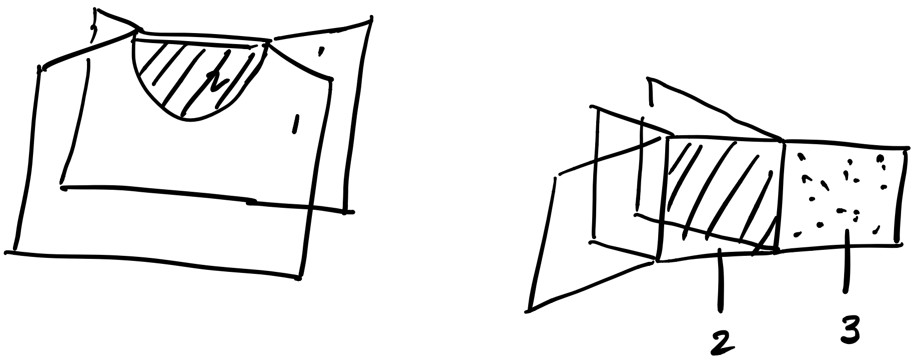
Ex:

T = 

cobordisms.

n>2 Foams

diagrams webs 



- allow decorations → dots. → q grading on your foams.

- yields a bigraded complex ⇝ take homology $\mathcal{H}_n$.

  q grading
  homological grading

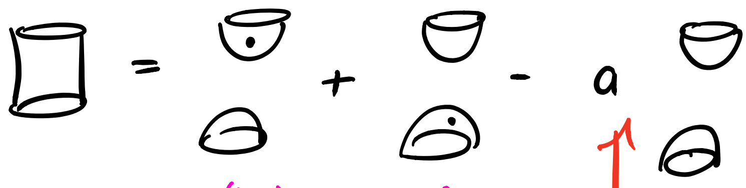
link diagram ⟶ Kom (Foam)
Kom (Cob) $/\sim$ ⟶ hom. cat of graded abelian rings

$\downarrow$ homology.

$\mathcal{H}_n$

Some of the relations that we need in $\text{Kom}(\text{Foam}) / \text{Kom}(\text{Cob}) / \sim$

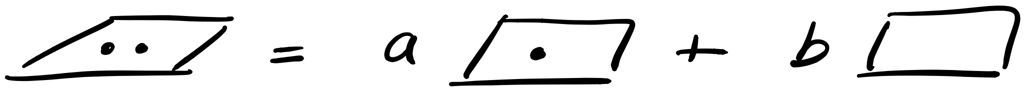
sl_2



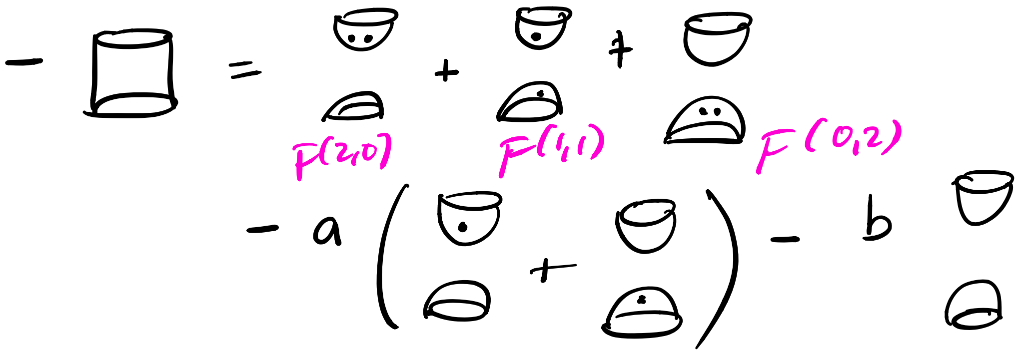
$F^{(1,0)}$ $F^{(0,1)}$ $F^{(0,0)}$

(neck cutting relation)

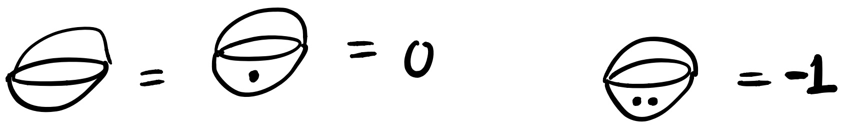
parameter corres choice of Frobenius alg.



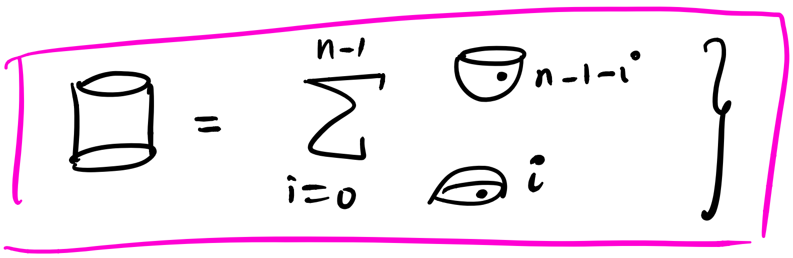
sl_3



$F^{(2,0)}$ $F^{(1,1)}$ $F^{(0,2)}$

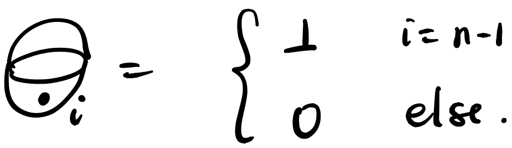


sl_n $(n \geq 4)$



comultiplication in $\Delta: A_n \to A_n \otimes_R A_n$

$\Delta(1)$ Frob. alg



evaluation of counit map.

② Ribbon concordance

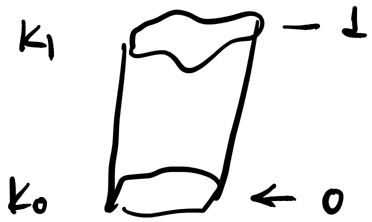
Conjecture (Gordon) Ribbon concordance gives you a partial order on knots in S^3

← ribbon

$$K_0 \underset{C_2}{\overset{C_1}{\rightleftarrows}} K_1 \quad \Rightarrow \quad K_1 = K_0$$

C_2 ← ribbon

Def: ① A <u>knot concordance</u> $K_0, K_1 \subseteq S^3$ is a smoothly embedded annulus in $S^3 \times [0,1]$ whose boundary $K_0 \times [0,1]$ and $K_1 \times [0,1]$



n component links

An <u>n-component link concordance</u> between L_0, L_1 is the disjoint union of n knot concordances btw the corresponding components.

A <u>ribbon concordance</u> C is a concordance whose projection onto the $[0,1]$ factor is a Morse function on C w/ only index 0 & 1 critical points

local minima

saddles.

Not a symmetric relation.

$$L_0 \xrightarrow{C_0} L_1 \quad \not\Rightarrow \quad L_1 \xrightarrow{\bar{C}_0} L_0$$

← need not be ribbon

<u>Zemke's original construction</u>

Let $C : L_0 \to L_1$ ribbon concordance $\bar{C} : L_1 \to L_0$

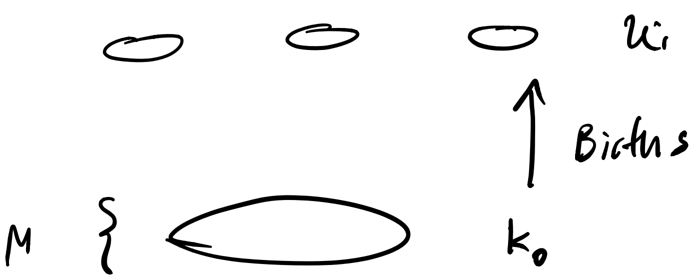
$\bar{C} \circ C : L_0 \to L_0$ obtained from the identity concordance of L_0 by "tubing in" unknotted, unlinked 2-spheres

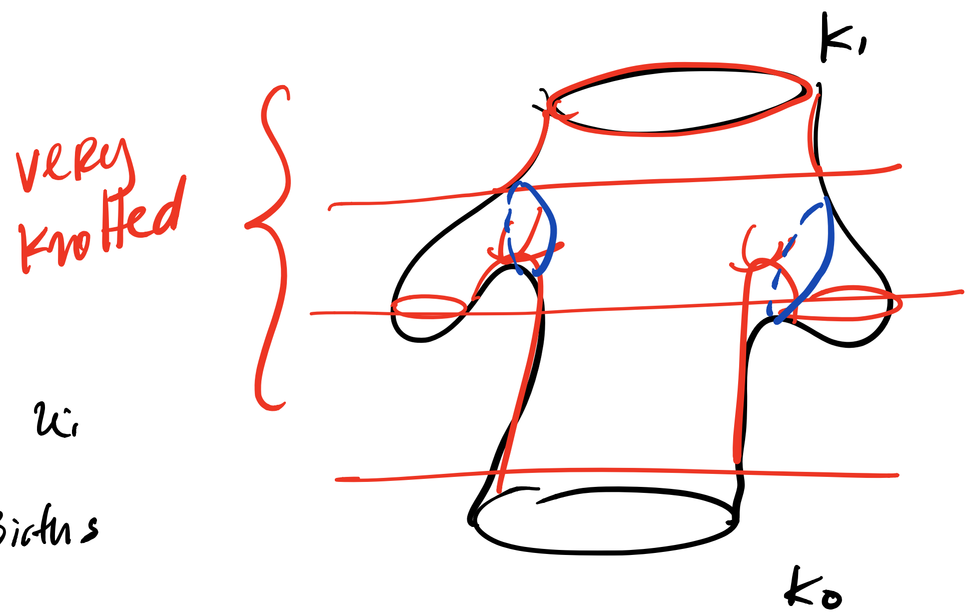
Step 1 pick a movie presentation of C w/ the following form.

(M1) n births: add n-unknots $U_1 \ldots U_n$ disjoint to K_0 from each other.

(M2) n saddles: B_i connecting U_i to K_0

(M3) A final isotopy to K_1





$\bar{C} \circ C$: n- births
n- saddles
n - dual saddles } cylinder (tubes) T_i
n - deaths

Zenko's procedure: • set T_i glued saddles.

• the births & deaths → spheres.

• set $D_i' = S_i \setminus \operatorname{int}(D_i)$ S_i - 2-sphere.

$\underbrace{}_{\text{puncture spheres}}$

$$\underline{\bar{C} \circ C} = \left((K_0 \times [0,1]) \setminus D_1^\circ \cup \dots \cup D_n^\circ \right) \cup \underline{(T_1 \cup \dots \cup T_n)} \cup \underline{(D_1' \cup \dots D_n')}$$

$\bar{C} \circ C \overset{\text{isotopic}}{\rightsquigarrow}$ identity concordance on K_0.

~ ok. instead of $S^3 \times [0,1]$

Lemma 1 $F \subseteq \mathbb{R}^3 \times [0,1]$ embedded cobordism.

from $L_0 \to L_1$

and $S^{(k)}$ unknotted 2-sphere in $\mathbb{R}^3 \times [0,1]$

unlinked from F w/ k dots.

a) $$\mathcal{H}_2(F \cup S) = \mathcal{H}_2(F) \otimes \cancel{\mathcal{H}_2(S)}^{\,0} = 0$$
$$\mathcal{H}_2(F \cup S^{(1)}) = \mathcal{H}_2(F)$$

b) $$\mathcal{H}_3(F \cup S^{(k)}) = \begin{cases} 0 & k \neq 2 \\ -\mathcal{H}_3(F) & k = 2 \end{cases}$$

c) $$\mathcal{H}_n(F \cup S^{(k)}) = \begin{cases} 0 & k \neq n-1 \\ \mathcal{H}_n(F) & k = n-1 \end{cases}$$

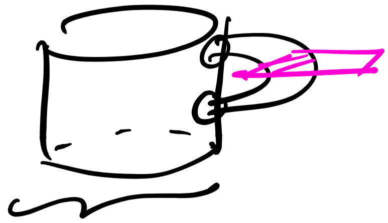
Pf: Functoriality of $\mathcal{H}_n$ and sphere relations.

Lemma: $F \subseteq \mathbb{R}^3 \times [0,1]$ embedded cob.

$L_0 \to L_1$

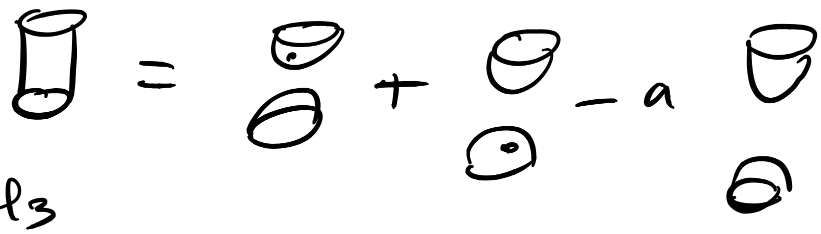
F' = attaching an embedded 3-dimensional 1 handle to F

Let $F^{(i,j)}$ be the cobordism obtained by neck cutting. Then



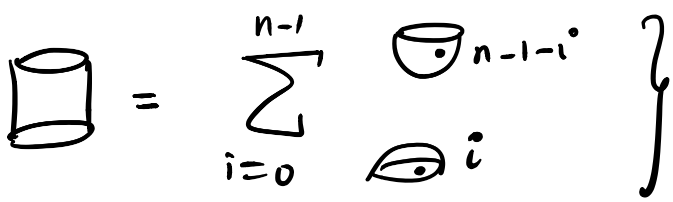
$F \cup S$.

a) $\mathcal{H}_2(F') = \mathcal{H}_2(F^{(1,0)}) + \mathcal{H}_2(F^{(0,1)}) - a\, \mathcal{H}_2(F)$



b) — $\underline{sl_3}$

c) $\mathcal{H}_n(F') = \sum_{i=0}^{n-1} \mathcal{H}_n(F^{(i,n-1-i)})$



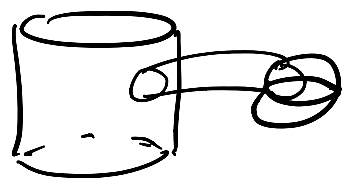
Pf: functoriality + neck cutting relations.

Proposition:

Let $D \subseteq \mathbb{R}^3 \times [0,1]$ embedded cob. $L_0 \to L_1$

S - 2sphere in $\mathbb{R}^3 \times [0,1]$ unknotted unliked from D.

D' = attaching 1 handle to D and S.



D

$$\mathcal{H}_n(D') = \mathcal{H}_n(D) \qquad \underline{n \geq 2}$$

Pf: directly follows from previous lemmas.

$$F = D \cup S \qquad F' = D'$$

Thm: Let C be a Ribbon concordance from $L_0 \to L_1$

Then the induced maps on sl_n-homology

$$\mathcal{H}_n(C): \mathcal{H}_n(L_0) \to \mathcal{H}_n(L_1)$$

are injective for $n \geq 2$

Pf: $D = \underbrace{\bar{C} \circ C}$

$\Updownarrow$

identity on $K_0 \times [0,1]$ w/ tubed in spheres

by prev. props $\Rightarrow$ $H_n(\bar{C} \circ C) = H_n(\bar{C}) \circ H_n(C)$

$= id_{H_n(L_0)}$

up to signs scalars.

$\Rightarrow$ $H_n(C)$ is injective and $H_n(\bar{C})$ is surjective

Since homological -q grading preserved under H_n.

$H_n^{i,j}(L_0) \hookrightarrow H_n^{i,j}(L_1)$. as a direct summand.

∎

$\{t\} \times [0,1]$