Bounded Modules of Direct Limit Lie Algebras

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Notation and Conventions

- Base field is $\mathbb{C}$;
- $\mathfrak{g}_n = \mathfrak{sl}(n+1), \mathfrak{sp}(2n), \mathfrak{so}(2n)$, or $\mathfrak{so}(2n+1)$, $V_n = \mathbb{C}^n$;
- $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{sp}(\infty)$, or $\mathfrak{o}(\infty)$ ($\mathfrak{g} = \varinjlim \mathfrak{g}_i$);
- $\mathfrak{h}_n$, $U_n = U(\mathfrak{g}_n)$: Cartan subalgebra and univ. env. alg. of $\mathfrak{g}_n$;
- $U = U(\mathfrak{g})$, $\mathfrak{h} = \varinjlim \mathfrak{h}_i$;
- $Q_{\mathfrak{g}_n}$, $Q_{\mathfrak{g}}$: the corresponding root lattices;
- For $\mu \in \mathbb{C}^{\mathbb{Z}_{>0}}$, $\mathrm{Int}(\mu)$, $\mathrm{Int}^+(\mu)$ are the subsets of $\mathbb{Z}_{>0}$ consisting of all i: $\mu_i \in \mathbb{Z}$, $\mu_i \in \mathbb{Z}_{\geq 0}$, respectively;
- $\mathcal{D}_n$: the algebra of polynomial diff. operators of $\mathbb{C}[x_1, ..., x_n]$;
- $\mathcal{D}(\infty)$: the subalgebra of $\mathrm{End}(\mathbb{C}[x_i]_{i \in \mathbb{Z}_{\geq 0}})$ generated by x_i , ∂_i.

Bounded and weight modules of $\mathfrak{g}$ and $\mathfrak{g}_n$

Definition

1. A module M of $\mathfrak{g}$ (or $\mathfrak{g}_n$) is a *weight module with finite weight multiplicities* if
   - $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$, where
     $M^\lambda := \{m \in M \mid h \cdot m = \lambda(h)m, \text{ for every } h \in \mathfrak{h}\}$.
   - $\dim M^\lambda < \infty$
2. M is *bounded* if it is a weight module and there is C such that $\dim M^\lambda < C$ for every λ

- Infinite-dimensional bounded modules exist for Lie algebras $\mathfrak{g}_n$ of type A and C only (Fernando).
- The simple bounded modules of $\mathfrak{g}_n$ were classified by O. Mathieu in 2000.
- The work of Mathieu combined with works of G. Benkart, D. Britten, S. Fernando, V. Futorny, F. Lemire, A. Joseph, and others, lead to the classification of all simple weight $\mathfrak{g}_n$-modules with finite weight multiplicities.

Problem: Classify all simple bounded modules of $\mathfrak{g}$.

Examples of bounded modules of $\mathfrak{g}_n = \mathfrak{sl}(n+1)$

- The map $e_{ij} \mapsto x_i \partial_j$ defines a homomorphism $U_n \to \mathcal{D}_{n+1}$;
- The $\mathcal{D}_{n+1}$-module $\mathcal{F}(\mu) = x^\mu \mathbb{C}[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$, $\mu \in \mathbb{C}^{n+1}$, has a $\mathfrak{g}_n$-submodule:

$$\mathcal{F}_{\mathfrak{sl}}(\mu) := \{x^\mu p \mid p \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}], \deg p = 0\}$$

- $\mathcal{F}_{\mathfrak{sl}}(\mu) = \mathcal{F}_{\mathfrak{sl}}(\mu')$ if and only if $\mu - \mu' \in Q_{\mathfrak{gl}(n+1)}$.
- The following is a submodule of $\mathcal{F}_{\mathfrak{sl}}(\mu)$:

$$V_{\mathfrak{sl}}(\mu) := \mathrm{Span}\{x^\lambda \mid \lambda - \mu \in Q_{\mathfrak{gl}(n+1)}, \mathrm{Int}^+(\mu) \subset \mathrm{Int}^+(\lambda)\}.$$

- $V_{\mathfrak{sl}}(\mu') \subset V_{\mathfrak{sl}}(\mu)$ if and only if $\mu - \mu' \in Q_{\mathfrak{gl}(n+1)}$ and $\mathrm{Int}^+(\mu) \subset \mathrm{Int}^+(\mu')$.
- For μ with $\mathrm{Int}^+(\mu) \subsetneq \mathrm{Int}(\mu)$, let

$$V_{\mathfrak{sl}}(\mu)^+ := \sum_{V_{\mathfrak{sl}}(\mu') \subsetneq V_{\mathfrak{sl}}(\mu)} V_{\mathfrak{sl}}(\mu').$$

- Finally, set $X_{\mathfrak{sl}}(\mu) := V_{\mathfrak{sl}}(\mu) / V_{\mathfrak{sl}}(\mu)^+$.

Classification of simple multiplicity-free $\mathfrak{sl}(n+1)$-modules

Theorem (Benkart-Britten-Lemire, 1997)

Every simple multiplicity-free (pointed) weight $\mathfrak{sl}(n+1)$-module is isomorphic either to $X_{\mathfrak{sl}}(\mu)$ for some $\mu \in \mathbb{C}^{n+1}$, or to $\Lambda^i(V_{n+1})$ for some i, $2 \leq i \leq n-1$. Furthermore,

$$Supp\, X_{\mathfrak{sl}}(\mu) = \{\mu' \in \mathbb{C}^{n+1} \mid \mu' \sim_{\mathfrak{sl}} \mu\},$$

where $\mu' \sim_{\mathfrak{sl}} \mu$ if $\mu - \mu' \in Q_{\mathfrak{gl}(n+1)}$ and $\mathrm{Int}^+(\mu) = \mathrm{Int}^+(\mu')$.

Lemma

If $\dim X_{\mathfrak{sl}}(\mu) < \infty$, then $X_{\mathfrak{sl}}(\mu) \simeq S^m(V_{n+1})$ for $m \geq 0$ or $X_{\mathfrak{sl}}(\mu) \simeq S^m(V_{n+1}^*)$ for $m \geq 0$.

Lemma

Let $n > 1$. $X_{\mathfrak{sl}}(\mu)$ and $X_{\mathfrak{sl}}(\mu')$ are isomorphic as $\mathfrak{sl}(n+1)$-modules if and only if either $\mu' \sim_{\mathfrak{sl}} \mu$ or $\{\mu, \mu'\} = \{0^{(n+1)}, (-1)^{(n+1)}\}$.

Results for $\mathfrak{g}_n = \mathfrak{sp}(2n)$

- We define a homomorphism $U_n \to \mathcal{D}_n$ by $e_{\varepsilon_i+\varepsilon_j} \mapsto x_i x_j$ if $i \neq j$, $e_{2\varepsilon_i} \mapsto \frac{1}{2}x_i^2$, $e_{-\varepsilon_i-\varepsilon_j} \mapsto -\partial_i\partial_j$ if $i \neq j$, $e_{-2\varepsilon_i} \mapsto -\frac{1}{2}\partial_i^2$, where $e_\alpha \in \mathfrak{g}_n^\alpha$ are appropriate nonzero vectors.
- For any $\mu \in \mathbb{C}^n$, we consider $\mathcal{F}(\mu)$ as a $\mathfrak{g}_n$-module through the homomorphism $U_n \to \mathcal{D}_n$. Then

$$\mathcal{F}_{\mathfrak{sp}}(\mu) := \{x^\mu p \mid p \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}], \deg p \in 2\mathbb{Z}\}$$

  is a multiplicity-free $\mathfrak{g}_n$-submodule of $\mathcal{F}(\mu)$.
- $\mathcal{F}_{\mathfrak{sp}}(\mu) = \mathcal{F}_{\mathfrak{sp}}(\mu')$ if and only if $\mu - \mu' \in Q_{\mathfrak{g}_n}$.
- Similarly to the definition of $X_{\mathfrak{sl}}(\mu)$, we define the $\mathfrak{g}_n$-module $X_{\mathfrak{sp}}(\mu)$.

Results for $\mathfrak{g}_n = \mathfrak{sp}(2n)$

Theorem

Let $n > 3$ and let be a simple multiplicity-free weight $\mathfrak{sp}(2n)$-module. If M is infinite dimensional, then M is isomorphic to $X_{\mathfrak{sp}}(\mu)$ for some $\mu \in \mathbb{C}^n$. If $\dim M < \infty$, then $M \simeq V_{2n}$ or $M \simeq \mathbb{C}$. Furthermore:

- $$\mathit{Supp}\, X_{\mathfrak{sp}}(\mu) = \{\mu' + \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right) \mid \mu' \sim_{\mathfrak{sp}} \mu\},$$

  where $\mu' \sim_{\mathfrak{sp}} \mu$ if $\mu - \mu' \in Q_{\mathfrak{sp}(2n)}$ and $\mathrm{Int}^+(\mu) = \mathrm{Int}^+(\mu')$.
- *The $\mathfrak{sp}(2n)$-modules $X_{\mathfrak{sp}}(\mu)$ and $X_{\mathfrak{sp}}(\mu')$ are isomorphic if and only if $\mu \sim_{\mathfrak{sp}} \mu'$.*

The case of $\mathfrak{g} = \mathfrak{sl}(\infty)$

From now on all results are from [G.-Penkov, 2019].

Lemma

Every simple bounded nonintegrable $\mathfrak{g}$-module is multiplicity free.

Lemma

Let M be a simple integrable bounded $\mathfrak{g}$-module. Then $M \simeq \varinjlim L_{\mathfrak{g}_n}(\lambda(n))$, where $L_{\mathfrak{g}_n}(\lambda(n))$ is the simple highest weight module of weight $\lambda(n)$, and $\lambda(n)$ is one of the following:

(i) $(1, 1, ..., 1, 0, 0, ..., 0)$, *where the number of 0's and 1's are both growing when $n \to \infty$;*

(ii) $(a_n, 0, 0, ..., 0)$,

(iii) $(0, 0, ..., 0, -a_n)$,

(iv) $(\mu_1, ..., \mu_k, 0, 0, ..., 0)$,

(v) $(0, 0, ..., 0, -\mu_k, ..., -\mu_1)$,

where k and $\mu_1, ..., \mu_k \in \mathbb{Z}_{>0}$ are fixed and such that $\mu_i - \mu_{i+1} \in \mathbb{Z}_{\geq 0}$, and $a_n \in \mathbb{Z}_{\geq 0}$ is monotonic with $\lim_{n\to\infty} a_n = \infty$.

Theorem

(i) *Every simple bounded nonintegrable* $\mathfrak{sl}(\infty)$*-module* M *is isomorphic to* $X_{\mathfrak{sl}}(\boldsymbol{\mu})$ *for some* $\boldsymbol{\mu} \in \mathbb{C}^{\mathbb{Z}_{>0}}$.

(ii) $X_{\mathfrak{sl}}(\boldsymbol{\mu}) \simeq X_{\mathfrak{sl}}(\boldsymbol{\mu}')$ *if and only if* $\boldsymbol{\mu} \sim_{\mathfrak{sl}} \boldsymbol{\mu}'$ *or* $\{\boldsymbol{\mu}, \boldsymbol{\mu}'\} = \{0^{(\infty)}, (-1)^{(\infty)}\}$.

(iii) *Supp* $X_{\mathfrak{sl}}(\boldsymbol{\mu}) = \{\lambda \mid \boldsymbol{\lambda} \sim_{\mathfrak{sl}} \boldsymbol{\mu}\}$.

Theorem

Every simple bounded integrable weight module of $\mathfrak{sl}(\infty)$ *is isomorphic to one of the following:*

(i) $\Lambda_{\mathbb{J}}^{\frac{\infty}{2}} V = \varinjlim \Lambda^{j_n}(V_{n+1})$ *for semi-inf. set* $\mathbb{J}$, $j_n = \#(\mathbb{J} \cap [1, n])$.

(ii) $S_{\mathbb{A}}^{\infty} V = \varinjlim S^{a_n}(V_{n+1})$ *for an inf. sequence* $\mathbb{A} = (a_1, a_2, \ldots)$.

(iii) $S_{\mathbb{A}}^{\infty} V_*$ *for an infinite sequence* $\mathbb{A}$,

(iv) $S^{\mu} V$ *for a partition* μ,

(v) $S^{\mu} V_*$ *for a partition* μ.

All isomorphisms between the above modules are: $\Lambda_{\mathbb{A}_1}^{\frac{\infty}{2}} V \simeq \Lambda_{\mathbb{A}_2}^{\frac{\infty}{2}} V$ *for* $\mathbb{A}_1 \approx \mathbb{A}_2$, $S_{\mathbb{A}_1}^{\infty} V \simeq S_{\mathbb{A}_2}^{\infty} V$ *and* $S_{\mathbb{A}_1}^{\infty} V_* \simeq S_{\mathbb{A}_2}^{\infty} V_*$ *for* $\mathbb{A}_1 \sim \mathbb{A}_2$, $S^{\emptyset} V = S^{\emptyset} V_* = \mathbb{C}$.

The case of $\mathfrak{g} = \mathfrak{sp}(\infty)$

Theorem

(i) *Every simple bounded nonintegrable* $\mathfrak{sp}(\infty)$*-module* M *is isomorphic to* $X_{\mathfrak{sp}}(\boldsymbol{\mu})$ *for some* $\boldsymbol{\mu} \in \mathbb{C}^{\mathbb{Z}_{>0}}$.

(ii) $X_{\mathfrak{sp}}(\boldsymbol{\mu}) \simeq X_{\mathfrak{sp}}(\boldsymbol{\mu}')$ *if and only if* $\boldsymbol{\mu} \sim_{\mathfrak{sp}} \boldsymbol{\mu}'$.

(iii) $\operatorname{Supp} X_{\mathfrak{sp}}(\boldsymbol{\mu}) = \{\lambda + \left(\frac{1}{2}\right)^{(\infty)} \mid \boldsymbol{\lambda} \sim_{\mathfrak{sp}} \boldsymbol{\mu}\}$. *In particular,* $\operatorname{Supp} X_{\mathfrak{sp}}(\boldsymbol{\mu})$ *determines* $X_{\mathfrak{sp}}(\boldsymbol{\mu})$ *up to isomorphism.*

Theorem

Let M *be a nontrivial simple bounded integrable weight* $\mathfrak{sp}(\infty)$*-module. Then* M *is isomorphic to the natural* $\mathfrak{sp}(\infty)$*-module* V.

Annihilators in the case $\mathfrak{g} = \mathfrak{sl}(\infty)$

For $x, y \in \mathbb{Z}_{\geq 0}$ and partitions λ, μ, denote by $I(x, y, \lambda, \mu)$ the annihilator of the $U(\mathfrak{sl}(\infty))$-module $(S^{\cdot}(V))^{\otimes x} \otimes (\Lambda^{\cdot}(V))^{\otimes y} \otimes S^{\lambda}V \otimes S^{\mu}V_*$. By a result of Penkov-Petukhov, $I(x, y, \lambda, \mu)$ is a primitive ideal and all primitive ideals of $U(\mathfrak{sl}(\infty))$ are of the form $I(x, y, \lambda, \mu)$.

Theorem

Let M be a simple bounded nonintegrable module of $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then $\mathrm{Ann}\, M = I(1, 0, \emptyset, \emptyset)$.

Theorem

Let $\mathfrak{g} = \mathfrak{sl}(\infty)$ and M be a nontrivial simple bounded integrable $\mathfrak{g}$-module.

(a) $\mathrm{Ann}\, M \simeq I(0, 1, \emptyset, \emptyset)$ *for* $M = \Lambda_{\mathbb{A}}^{\frac{\infty}{2}} V$;

(b) $\mathrm{Ann}\, M \simeq I(1, 0, \emptyset, \emptyset)$ *for* $M = S_{\mathbb{A}}^{\infty} V$ *or* $M = S_{\mathbb{A}}^{\infty} V_*$;

(c) $\mathrm{Ann}\, M \simeq I(0, 0, \lambda, \emptyset)$ *for* $M = S^{\lambda} V$;

(d) $\mathrm{Ann}\, M \simeq I(0, 0, \emptyset, \mu)$ *for* $M = S^{\mu} V_*$.

Future work: Lie superalgebras

For $\mathbb{A} \subset \mathbb{Z}_{>0}$ we may define two spinor type modules: $\mathcal{S}^B_{\mathbb{A}}$ (weight module of type B), and $\mathcal{S}^D_{\mathbb{A}}$ (weight module of type B).

Theorem

Let M be a simple weight module which is a direct limit of spinor modules of $\mathfrak{o}(2n+1)$ or $\mathfrak{o}(2n)$ for $n \to \infty$. Then $M \simeq \mathcal{S}^B_{\mathbb{A}}$ or $M \simeq \mathcal{S}^D_{\mathbb{A}}$ for some $\mathbb{A} \subset \mathbb{Z}_{>0}$. Furthermore, $\mathcal{S}^B_{\mathbb{A}_1} \simeq \mathcal{S}^B_{\mathbb{A}_2}$ if and only if $\mathbb{A}_1 \sim \mathbb{A}_2$.

Future directions:

- We expect that the modules $\mathcal{F}_{sp}(\mu)$ will be injective, and we will have equivalence of categories $\mathfrak{sp}(\infty)$-bounded modules and $\mathcal{D}(\infty)$-weight modules.
- The simple bounded modules over $\mathfrak{sp}(\infty)$ and $\mathfrak{o}(\infty)$ "combine" nicely over $\mathfrak{g} = \mathfrak{osp}(\infty|\infty)$. Namely, if $\mathfrak{h}$ is of type D, then every simple nontrivial bounded $\mathfrak{g}$-module is isomorphic to $X^D(\boldsymbol{\mu}, \mathbb{A})$ for some $\boldsymbol{\mu} \in \mathbb{C}^{\mathbb{Z}_{>0}}$ and $\mathbb{A} \subset \mathbb{Z}_{>0}$, or to the natural $\mathfrak{g}$-module. Note $X^D(\boldsymbol{\mu}, \mathbb{A})_{\bar{0}} \simeq \mathcal{S}^D_{\mathbb{A}} \boxtimes X_{\mathfrak{sp}}(\boldsymbol{\mu})$. (joint with I. Penkov and V. Serganova).

Thank you!