Springer fibers, rank varieties, and generalized coinvariant rings

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Geometric representations

This talk is about graded rings with S_n-actions, particularly ones with connections to geometry.

Primary example:

Geometric representations

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Primary example:

1) The cohomology of the complete flag variety $H^*(\mathrm{Fl}(n); \mathbb{Q})$

2) The coinvariant ring $R_n := \mathbb{Q}[x_1, \ldots, x_n]/I_n$

3) Coordinate ring of the scheme of "nilpotent" diagonal matrices $\mathbb{Q}[\mathcal{N} \cap \mathfrak{t}]$

$$H^*(\mathrm{Fl}(n); \mathbb{Q}) \cong R_n \cong \mathbb{Q}[\mathcal{N} \cap \mathfrak{t}]$$

Where we are going

Ring	S_n-mod structure	Cohomology	Coord. ring
R_n	Regular rep $\mathbb{Q}S_n$	$H^*(\mathrm{Fl}(n);\mathbb{Q})$	$\mathbb{Q}[\mathcal{N}\cap\mathfrak{t}]$
R_λ	$\mathbb{Q}\,S_n/$Young subgrp	$H^*(\mathcal{B}_\lambda;\mathbb{Q})$	$\mathbb{Q}[\overline{O}_{\lambda'}\cap\mathfrak{t}]$
$R_{n,k}$	OSPs	$H^*(X_{n,k};\mathbb{Q})$	?

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$R_{n,k}$	OSPs	$H^*(X_{n,k};\mathbb{Q})$	$\mathbb{Q}[\overline{O}_{n,(k)}\cap\mathfrak{t}]$
$R_{n,\lambda}$	(n,λ)-OSPs	?	$\mathbb{Q}[\overline{O}_{n,\lambda'}\cap\mathfrak{t}]$

Partial connection
At the end if time

$\mathrm{Fl}(n)$ and coinvariant rings R_n

Cohomology of $\mathrm{Fl}(n)$

A *complete flag* of subspaces of $\mathbb{C}^n$ is a chain

$$V_\bullet = (0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$$

such that $\dim_{\mathbb{C}} V_i = i$.

The *complete flag variety* $\mathrm{Fl}(n)$ is the space of all complete flags in $\mathbb{C}^n$.

$e_2(\underline{x}_3) = x_1x_2 + x_1x_3 + x_2x_3$

- $\mathbf{x}_n = \{x_1, \ldots, x_n\}$.
- $e_d(\mathbf{x}_n) =$ sum of degree d square free monomials in $\mathbf{x}_n$.

Theorem (Borel, 1953)

$$H^*(\mathrm{Fl}(n);\mathbb{Q}) \cong \frac{\mathbb{Q}[\mathbf{x}_n]}{\langle e_1, \ldots, e_n \rangle} =: R_n$$

$x_i = -c_1(V_i/V_{i-1})$

Coinvariant rings R_n

The quotient ring

$$R_n := \frac{\mathbb{Q}[\mathbf{x}_n]}{\langle e_1, e_2, \ldots, e_n \rangle}$$

is called the *coinvariant ring*.

Combinatorial/Rep theoretic properties:

- Dimension: $\dim_{\mathbb{Q}}(R_n) = n! = 2 \cdot 3 \cdots n$
- S_n-module: $R_n \cong \mathbb{Q}S_n$ as S_n-modules
- Hilbert series:
  $$\mathrm{Hilb}(R_n; q) = \overbrace{\underbrace{(1+q)}_{[2]_q}\underbrace{(1+q+q^2)}_{[3]_q} \cdots \underbrace{(1+\cdots+q^{n-1})}_{[n]_q}}^{[n]!_q}$$

Scheme of "nilpotent" diagonal matrices

Let $\mathcal{N} = \{X \in \mathfrak{gl}_n : X^n = 0\}$ the nilpotent cone

Observe: $I(\mathcal{N}) \neq \langle$entries of $X^n\rangle$. $\to$ $X \in \mathcal{N} \Rightarrow tr(X) = 0$

$$\det(tI_n - X) = \sum_{i=0}^{n} \underline{\sigma_i(X)} t^{n-i}$$

X is nilpotent iff $\sigma_i(X) = 0$ for all $i > 0$.

$I(\mathcal{N}) = \langle \sigma_i(X) \mid i > 0 \rangle$

$X = \mathrm{diag}(x_1, \ldots, x_n)$

$\sigma_i(X) = e_i(x_1, \ldots, x_n)$

Theorem (Kostant, 1963)

Let $\mathfrak{t}$ be $n \times n$ diagonal matrices. As graded rings,

$$R_n \cong \mathbb{Q}[\underline{\mathcal{N} \cap \mathfrak{t}}].$$

$\hookrightarrow$ Scheme-th. intersection supported on $\{0\}$

Set-theoretic intersection: $\mathcal{N} \cap \mathfrak{t} = \{0\}$

Ring	S_n-mod structure	Cohomology	Coord. ring
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$R_{n,k}$	OSPs	$H^*(X_{n,k};\mathbb{Q})$	$\mathbb{Q}[\overline{O}_{n,(k)} \cap \mathfrak{t}]$
$R_{n,\lambda}$	(n,λ)-OSPs	?	$\mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]$

Cohomology of a Springer Fiber R_λ

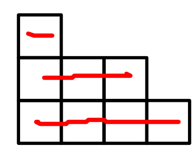
Partitions

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ an *integer partition* of n, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and $\sum_i \lambda_i = n$.
We say $\ell =: \ell(\lambda)$ is the *length* of λ.

Young diagram: Rows of boxes given by the parts of λ

Example:

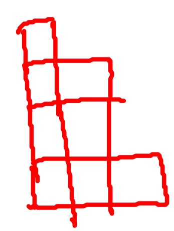
$$\lambda = (4, 3, 1) \quad \longleftrightarrow$$



The *conjugate* of λ, denoted by λ', records the sizes of the columns of the Young diagram of λ.

Example:

$$\lambda' = (3, 2, 2, 1) \quad \longleftrightarrow$$



Springer fiber $\mathcal{B}_\lambda$

Given λ a partition of n, let

$$O_\lambda = \{X \in \mathcal{N} \text{ Jordan type } \lambda\}$$

Ex $\lambda = (3,2)$

$$X = \left[\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ \hline & & & 0 & 1 \\ & & & & 0 \end{array}\right]$$

Given $X \in O_\lambda$, the *Springer fiber* associated to X is

$$\mathcal{B}_X := \{V_\bullet \in \mathrm{Fl}(n) \,:\, \underline{XV_i \subseteq V_i} \text{ for all } i\}.$$

$$T^*\mathrm{Fl}(n) = \{(V_\bullet, X) \in \mathrm{Fl}(n) \times \mathcal{N} \mid XV_i \subseteq V_i \ \forall i\}$$

$$\begin{tikzcd} T^*\mathrm{Fl}(n) \arrow[d, "\mu"] \\ \mathcal{N} \end{tikzcd}$$

Springer resolution of $\mathcal{N}$

$$\mathcal{B}_X = \mu^{-1}(X)$$

Let $\mathcal{B}_\lambda := \mathcal{B}_X$ for any $X \in O_\lambda$.

S_n-module structure of $H^*(\mathcal{B}_\lambda;\mathbb{Q})$.

Springer was the first to construct an S_n-action on $H^*(\mathcal{B}_\lambda;\mathbb{Q})$. He proved

$$H^*(\mathcal{B}_\lambda;\mathbb{Q}) \cong \mathbb{Q}\left(\frac{S_n}{S_{\lambda_1}\times S_{\lambda_2}\times\cdots\times S_{\lambda_m}}\right)$$

$\to$ Young subgrp

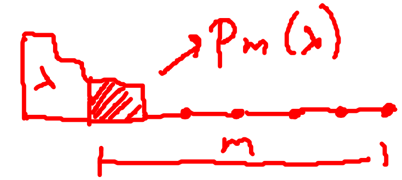
$$H^{\mathsf{top}}(\mathcal{B}_\lambda;\mathbb{Q}) \cong S^\lambda \text{ the irreducible } S_n\text{-module}$$

Springer fibers provide a geometric way of constructing the irreducible S_n-modules.

Graded S_n-module structure characterized by Hotta-Springer, later Garsia-Procesi.

Explicit presentation of $H^*(\mathcal{B}_\lambda;\mathbb{Q})$

Write $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0)$.



For $1 \leq m \leq n$, let $p_m(\lambda) := \underbrace{\lambda'_n + \lambda'_{n-1} + \cdots + \lambda'_{n-m+1}}_{\text{sum of last } m \text{ entries of } \lambda'}$.

The *Tanisaki ideal* I_λ is

$$I_\lambda := \langle e_d(S) \; : \; S \subseteq \mathbf{x}_n, \;\; d > |S| - p_{|S|}(\lambda) \rangle,$$

$$R_\lambda := \mathbb{Q}[\mathbf{x}_n]/I_\lambda.$$

Theorem (De Concini-Procesi '81, Tanisaki '82)

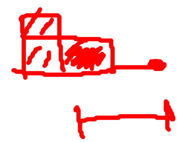
$$H^*(\mathcal{B}_\lambda;\mathbb{Q}) \cong R_\lambda$$

Example of the ring R_λ

depends on $|S|$, not S

$$I_\lambda = \langle e_d(S) \,:\, S \subseteq \mathbf{x}_n,\ |S| \geq d > |S| - p_{|S|}(\lambda) \rangle.$$

Example: $n = 3$, $\lambda = (2,1)$



Generators of $I_{(2,1)}$:

- $|S| = 3$: $p_3(\lambda) = 3$, $3 \geq d > 0$, $\boxed{e_1(x_1,x_2,x_3),\ e_2(x_1,x_2,x_3),\ e_3(x_1,x_2,x_3)}$
- $|S| = 2$: $p_2(\lambda) = 1$, $2 \geq d > 1$, $\boxed{e_2(x_1,x_2) = x_1x_2,\ e_2(x_1,x_3) = x_1x_3,\ e_2(x_2,x_3) = x_2x_3}$
- $|S| = 1$: $p_1(\lambda) = 0$, $\emptyset$

Observe: I_λ is closed under the S_n-action.

"Diagonal" nilpotents for R_λ

$$O_\lambda = \{X \in \mathcal{N} \text{ Jordan type } \lambda\}$$
$$= \{X \in \mathfrak{gl}_n \ : \ \mathrm{rk}(X^i) = p_{n-i}(\lambda) \text{ for all } i\}$$

$$\overline{O}_\lambda = \{X \in \mathfrak{gl}_n \ : \ \mathrm{rk}(X^i) \leq p_{n-i}(\lambda) \text{ for all } i\}.$$

<u>Ex</u> $\lambda = (3,2)$

$$X = \left[\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ \hline & & & 0 & 1 \\ & & & & 0 \end{array}\right], \quad X^2 = \left[\begin{array}{ccc|cc} 0 & 0 & 1 & & \\ & 0 & 0 & & \\ & & 0 & & \\ \hline & & & 0 & 0 \\ & & & & 0 \end{array}\right], \quad X^3 = 0$$

$\mathrm{rk}\, X = \underline{3}$ $\qquad$ $\mathrm{rk}\, X^2 = \underline{1}$ $\qquad$ $\mathrm{rk}\, X^3 = \underline{0}$

Theorem (De Concini-Procesi, 1981)

As graded rings,

$$H^*(\mathcal{B}_\lambda; \mathbb{Q}) \cong R_\lambda \cong \mathbb{Q}[\overline{O}_{\lambda'} \cap \mathfrak{t}].$$

<u>Ex</u> $\lambda = (1^n)$
$\overline{O}_{\lambda'} = \mathcal{N}$

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Generalized coinvariant rings $R_{n,k}$

The rings $R_{n,k}$

Generalized coinvariant rings of Haglund-Rhoades-Shimozono:

Given $k \leq n$,

$$I_{n,k} := \langle \underbrace{e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n)}_{\text{top } k \text{ degree } e\text{'s}} \rangle + \langle \underline{x_1^k, \ldots, x_n^k} \rangle$$

$$R_{n,k} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k}.$$

Ex $k=n$

$I_{n,n} = \langle e_n, .., e_1 \rangle + \langle x_1^n, .., x_n^n \rangle = \langle e_n, .., e_1 \rangle = I_n$

HRS were motivated by the Delta Conjecture from Algebraic Combinatorics. They proved $R_{n,k}$ gives a rep-theoretic interpretation of the $t = 0$ case of this conjecture.

S_n-mod structure of $R_{n,k}$

$\mathcal{OP}_{n,k}$ = Ordered set partitions of $\{1, \ldots, n\}$ into k nonempty blocks

S_n acts on $\mathcal{OP}_{n,k}$ by permuting $1, 2, \ldots, n$.

Example: $\tau = (2,3)$

$$\mathcal{OP}_{3,2} = \{(12|3), (13|2), (23|1), (1|23), (2|13), (3|12)\}$$

Theorem (HRS, 2018)

$$R_{n,k} \cong \mathbb{Q}\,\mathcal{OP}_{n,k} \text{ as } S_n\text{-modules}$$

Geometry of $R_{n,k}$

Pawlowski and Rhoades found a connection to *spanning line arrangements*.

$n=k$

$X_{n,n} \cong GL_n/T \simeq Fl(n)$

Let $X_{n,k}$ be

$S_n \circlearrowleft$

$$\{(L_1, L_2, \ldots, L_n) : L_i \in \mathbb{P}^{k-1},\ L_1 + L_2 + \cdots + L_n = \mathbb{C}^k\}$$

$X_{n,k}$ is a Zariski open subset of $(\mathbb{P}^{k-1})^n$. It is a smooth *noncompact* manifold.

Theorem (Pawlowski-Rhoades, 2019)

$R_{n,k} \cong H^*(X_{n,k};\mathbb{Q})$ as graded rings.

Summary

Ring	S_n-mod structure	Cohomology	Coord. ring
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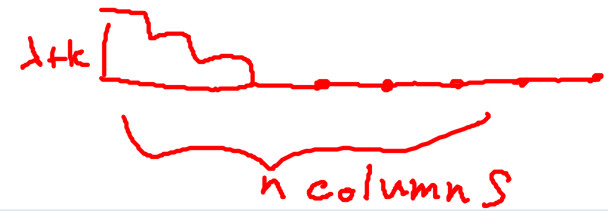
$R_{n,k}$	OSPs	$H^*(X_{n,k};\mathbb{Q})$	$\mathbb{Q}[\overline{O}_{n,(k)} \cap \mathfrak{t}]$
$R_{n,\lambda}$	(n,λ)-OSPs	?	$\mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]$

New family of rings $R_{n,\lambda}$

The rings $R_{n,\lambda}$

Let $k \le n$ and $\lambda \vdash k$.

Write $\lambda' = (\lambda'_1 \ge \lambda'_2 \ge \cdots \ge \lambda'_n \ge 0)$



Let $p^n_m(\lambda) = \lambda'_n + \cdots + \lambda'_{n-m+1}$.

$$I_{n,\lambda} := \langle e_d(S) \,:\, S \subseteq \mathbf{x}_n,\, d > |S| - p^n_{|S|}(\lambda) \rangle,$$

smaller

$$\underline{\underline{R_{n,\lambda}}} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$$

infinite dim'l $\mathbb{Q}$-v.s.

The rings $R_{n,\lambda}$

Let $k \leq n$ and $\lambda \vdash k$.

Write $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0)$

Let $p^n_m(\lambda) = \lambda'_n + \cdots + \lambda'_{n-m+1}$.

$$I_{n,\lambda} := \langle e_d(S) \,:\, S \subseteq \mathbf{x}_n,\, d > |S| - p^n_{|S|}(\lambda) \rangle,$$
$$R_{n,\lambda} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$$

For $s \geq \ell(\lambda)$, also define

$$I_{n,\lambda,s} := I_{n,\lambda} + \langle x_1^s, \ldots, x_n^s \rangle,$$
$$R_{n,\lambda,s} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda,s}$$

Ex1 When $n=k$, $R_{n,\lambda,s} = R_\lambda$ for any s

Ex2 When $\lambda = (1^k)$ and $s=k$, $R_{n,(1^k),k} = R_{n,k}$

Rank varieties

(Eisenbud-Saltman) Given $k \leq n$ and $\lambda \vdash k$, let

$$\overline{O}_{n,\lambda} = \{X \in \mathfrak{gl}_n \ : \ \mathrm{rk}(X^i) \leq (n-k) + p^n_{n-i}(\lambda) \text{ for all } i\}.$$

$$X = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \qquad \begin{array}{l} A \in \overline{O}_\lambda \\ B \in \mathfrak{gl}_{n-k} \end{array}$$

Eisenbud-Saltman conjectured an explicit generating set for $I(\overline{O}_{n,\lambda})$, which Weyman (1989) proved.

Corollary (G., 2020)

As graded rings and graded S_n-modules,

$$R_{n,\lambda} \cong \mathbb{Q}[\overline{O}_{n,\lambda'} \cap \mathfrak{t}]$$

Main results

- (Ungraded) S_n-module structure of $R_{n,\lambda}$ and $R_{n,\lambda,s}$
- Hilbert series formulas for $R_{n,\lambda}$ and $R_{n,\lambda,s}$
- Graded S_n-module structure of $R_{n,\lambda}$ and $R_{n,\lambda,s}$
  - Monomial expansion (HHL-type formula)
  - Hall-Littlewood expansion

Theme: Formula for $R_{n,\lambda}$ is the "limit" of formula for $R_{n,\lambda,s}$ as $s \to \infty$.

(Ungraded) S_n-module structure

Let $\mathcal{OP}_{n,\lambda,s}$ be the set of length $\underline{s}$ ordered set partitions of $[n]$, $(B_1|\cdots|B_s)$ such $|B_i| \geq \lambda_i$.   (Allow $B_i = \emptyset$ if $i > \ell(\lambda)$)

Example: $n = 4$, $\lambda = (2,1)$, $s = 2$

$$(\underbrace{123}_{\geq 2}|\underbrace{4}_{\geq 1}),\ (124|3),\ (134|2),\ (234|1)$$

$$(12|34),\ (13|24),\ (14|23),\ (23|14),\ (24|13),\ (34|12)$$

Theorem (G., 2020)

$R_{n,\lambda,s} \cong \mathbb{Q}\,\mathcal{OP}_{n,\lambda,s}$ as S_n-modules.

$R_{n,\lambda} \cong \mathbb{Q}\,\mathcal{OP}_{n,\lambda,\infty}$ as S_n-modules.

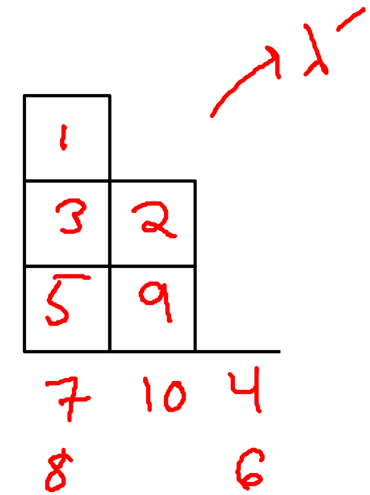
→ inf. many copies of some irred.

Hilbert series of $R_{n,\lambda,s}$

To each $\sigma \in \mathcal{OP}_{n,\lambda,s}$, associate a *standard extended column-increasing labeling (SECI)* of λ':

Example: $n = \overset{10}{\cancel{7}}$, $\lambda = (3, 2)$, $s = 3$

$$\sigma = (1, 3, 5, 7, 8 \,|\, 2, 9, 10 \,|4, 6) \longleftrightarrow$$



Hilbert series of $R_{n,\lambda,s}$

To each $\sigma \in \mathcal{OP}_{n,\lambda,s}$, associate a *standard extended column-increasing labeling (SECI)* of λ':

Example: $n = 7$, $\lambda = (3,2)$, $s = 3$

$$\sigma = (1,3,5,7,8 \,|\, 2,9,10 \,|\, 4,6) \longleftrightarrow \quad \varphi = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 3 & 2 & \\ \hline 5 & 9 & \\ \hline \boxed{7} & \boxed{10} & \boxed{4} \\ \hline \boxed{8} & & \boxed{6} \\ \hline \end{array}$$

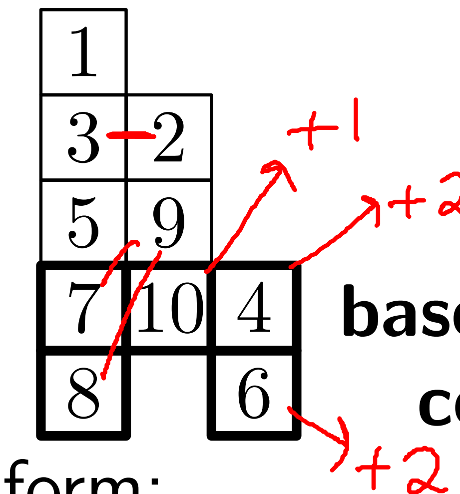
basement cells

Hilbert series of $R_{n,\lambda,s}$

To each $\sigma \in \mathcal{OP}_{n,\lambda,s}$, associate a *standard extended column-increasing labeling (SECI)* of λ':

Example: $n = 7$, $\lambda = (3,2)$, $s = 3$

$\sigma = (1,3,5,7,8\,|\,2,9,10\,|\,4,6) \quad \longleftrightarrow \quad \varphi =$

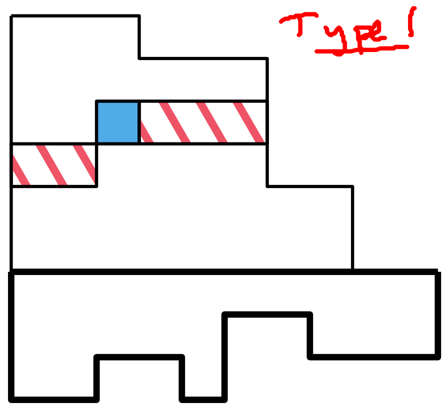


basement cells

An <u>*inversion*</u> is a pair of labels $\boxed{b} > \boxed{a}$ of the form:

Type 1

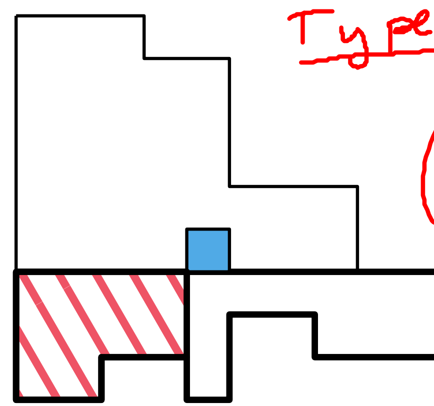
1



or

Type 2

2



inv(φ)= 3 + 5 = 8

Let $\mathrm{inv}(\varphi) = \#$ inversions

$+ \sum_i (i-1)\,\#$(basement cells in col. i).

Hilbert series of $R_{n,\lambda,s}$

The statistic inv is a variation on the coinv statistic of Rhoades-Yu-Zhao.

Theorem (Rhoades-Yu-Zhao, 2020)

$$\mathrm{Hilb}(R_{n,\lambda,s};q) = \sum_{\varphi \in \mathrm{SECI}_{n,\lambda,s}} q^{\mathrm{inv}(\varphi)}.$$

There is also another statistic dinv which gives a formula for the Hilbert series.

Theorem (G., 2020)

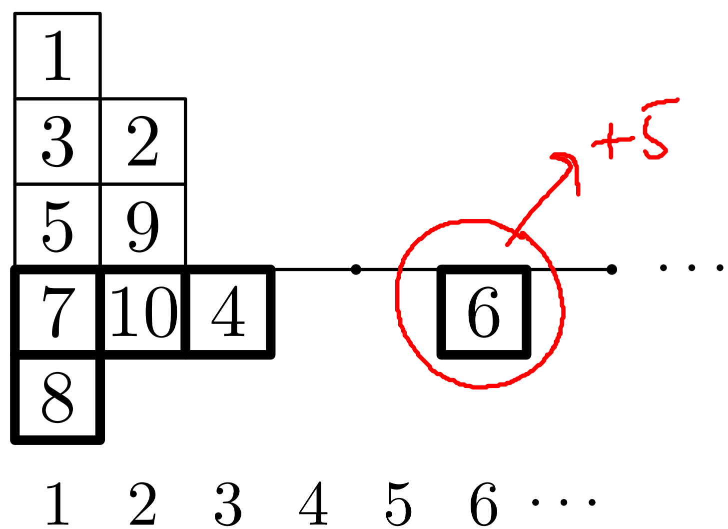
$$\mathrm{Hilb}(R_{n,\lambda,s};q) = \sum_{\varphi \in \mathrm{SECI}_{n,\lambda,s}} q^{\mathrm{dinv}(\varphi)}.$$

Hilbert series of $R_{n,\lambda}$

To get a formula for $R_{n,\lambda}$, we let $s \to \infty$, meaning we allow a basement cells in any column $i \geq 0$. Let $\mathrm{SECI}_{n,\lambda,\infty}$ be the resulting set of standard extended column-increasing labelings.

Example: $n = 10$, $\lambda = (3, 2)$

$\varphi =$



Theorem (G., 2020)

$$\mathrm{Hilb}(R_{n,\lambda}; q) = \sum_{\varphi \in \mathrm{SECI}_{n,\lambda,\infty}} q^{\mathrm{inv}(\varphi)}.$$

Graded S_n-module structure of $R_{n,\lambda}$

Graded Frobenius characteristic

A *symmetric function* is a formal power series in $\mathbf{x} = \{x_1, x_2, \ldots\}$ which is symmetric in the variables.

The *Frobenius characteristic* is the map from S_n-modules to symmetric functions such that

$$\mathrm{Frob}(S^\lambda) = s_\lambda(\mathbf{x})$$

Schur function

and extended "linearly".

To each graded S_n-module $V = \bigoplus_{i \geq 0} V_i$, we associate its *graded Frobenius characteristic*,

$$\mathrm{Frob}(V; q) := \sum_{i \geq 0} \mathrm{Frob}(V_i) q^i.$$

Graded Frobenius of R_λ

Let $\mathrm{Sym}(\mathbb{Q}(q))$ be the algebra of symmetric functions, with coefficients in $\mathbb{Q}(q)$.

It has a basis given by *Hall-Littlewood symmetric functions* $Q'_\lambda(\mathbf{x};q)$. $= H_\lambda(\underline{x};q)$

Theorem (Hotta-Springer '77, Garsia-Procesi '92)

$$\mathrm{Frob}(R_\lambda;q) = \mathrm{rev}_q[Q'_\lambda(x;q)]$$

Garsia and Procesi reproved this result of Hotta-Springer using elementary methods starting from Tanisaki's ideals.

Graded Frobenius of $R_{n,k}$

For $\mu \vdash n$, let $m_i(\mu) =$ number of parts of μ of size i.

- q-multinomial coefficient:

$$\begin{bmatrix} k \\ a_1, \ldots, a_n \end{bmatrix}_q := \frac{[k]_q!}{[a_1]_q! \cdots [a_n]_q!}$$

Theorem (HRS, 2018)

$\mathrm{Frob}(R_{n,k}; q)$ has the following formula

$$\mathrm{rev}_q \left[\sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{\sum_i (i-1)(\mu_i - 1)} \begin{bmatrix} k \\ m_1(\mu), \ldots, m_n(\mu) \end{bmatrix}_q Q'_\mu(x;q) \right].$$

Graded Frobenius of $R_{n,\lambda,s}$

- q-binomial coefficient:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q := \frac{[a]_q!}{[b]_q![a-b]_q!}$$

- $n(\mu,\lambda) = \sum_{i \geq 1} \binom{\mu_i' - \lambda_i'}{2}$

Theorem (G., 2020)

$\mathrm{Frob}(R_{n,\lambda,s}; q)$ has the following expansion

$$\mathrm{rev}_q \left[\sum_{\substack{\mu \vdash n \\ \ell(\mu)=\ell(\lambda) \\ \mu_i \geq \lambda_i \forall i}} \underline{q^{n(\mu,\lambda)}} \underline{\prod_{i \geq 0} \begin{bmatrix} \mu_i' - \lambda_{i+1}' \\ \mu_i' - \mu_{i+1}' \end{bmatrix}_q} \underline{Q_\mu'(x;q)} \right].$$

$\mu_0' := s$

Graded Frobenius of $R_{n,\lambda}$

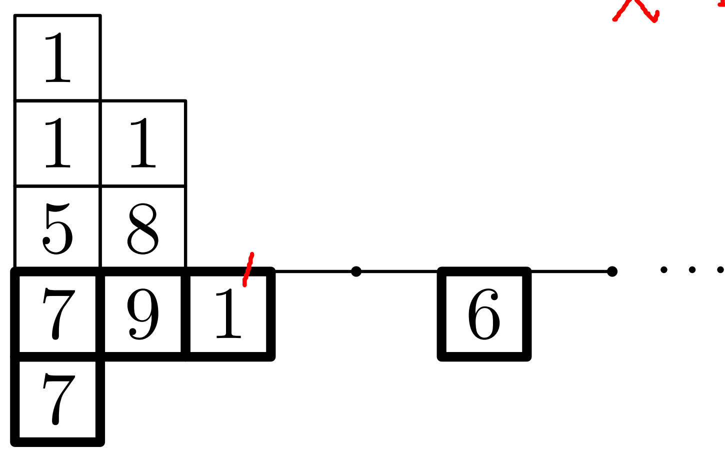
Let $\mathrm{ECI}_{n,\lambda,\infty}$ be the set of *extended column-increasing fillings* of λ'.

Define the same *inversion* statistic on these non-standard fillings.

Example: $n = 10$, $\lambda = (3,2)$

$x^{\varphi} = x_1^4 x_5 x_6 x_7^2 x_8 x_9$

$\varphi =$



Theorem (G., 2020)

$$\mathrm{Frob}(R_{n,\lambda}; q) = \sum_{\varphi \in \mathrm{ECI}_{n,\lambda,\infty}} q^{\mathrm{inv}(\varphi)} \mathbf{x}^{\varphi}.$$

Further Directions

1) Is there a "generalized Springer fiber" $\mathcal{B}_{n,\lambda,s}$ whose rational cohomology ring is $R_{n,\lambda,s}$? Or for $R_{n,\lambda}$?

2) Haglund-Rhoades-Shimozono related $\mathrm{Frob}(R_{n,k}; q)$ to the operators Δ' coming from Haiman's work on the $n!$-conjecture. Is $\mathrm{Frob}_q(R_{n,\lambda})$ also related to Δ' operators?

Thanks for listening!

Extras:

Partial connection to Springer fibers

Rank varieties

(Eisenbud-Saltman) Given $k \leq n$ and $\lambda \vdash k$, let

$$\overline{O}_{n,\lambda} = \{X \in \mathfrak{gl}_n \, : \, \mathrm{rk}(X^i) \leq (n-k) + p^n_{n-i}(\lambda) \text{ for all } i\}.$$

Weyman proved an explicit generating set for $I(\overline{O}_{n,\lambda})$, which was conjectured by Eisenbud-Saltman.

$$\bigwedge^m (tI_n - X) = [m \times m \text{ minors of } tI_n - X]$$

$$J^m_d = \langle t^{m-d}\text{-coefficients of entries of } \bigwedge^m (tI_n - X)\rangle$$

Theorem (Weyman, 1989)

$$I(\overline{O}_{n,\lambda}) = \sum_{d > m - p^n_m(\lambda)} J^m_d$$

Monomial basis

Example

$$k = 7 \qquad \lambda = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$
$$n = 9$$

An (n, λ, ∞)-*staircase* is a shuffle of the compositions

$$(0, 1, 2), \quad (0, 1), \quad (0), \quad (0), \quad (\infty)^{n-k}$$

Let $\mathcal{A}_{n,\lambda,\infty} =$
$$\{x_1^{a_1} \cdots x_n^{a_n} \ : \ (a_1, \ldots, a_n) \leq \text{ some } (n, \lambda, \infty)\text{-staircase}\}.$$

Theorem (G., 2020)

$\mathcal{A}_{n,\lambda,\infty}$ is a basis of $R_{n,\lambda}$.