Self-dual puzzles in Schubert calculus branching

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UC Davis
Algebra and Discrete Mathematics Seminar
June 1, 2020

Table of Contents

Grassmannians I

We are interested in the cohomology pullback of

$$Gr(k,2n) := \{V \subseteq \mathbb{C}^{2n} \mid \dim V = k\} \cong GL_{2n}/P$$

$$\Big\uparrow \iota$$

$$SpGr(k,2n) := \{V \subseteq \mathbb{C}^{2n} \mid \dim V = k, V \subseteq V^{\perp}\} \cong Sp_{2n}/(P \cap Sp_{2n})$$

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General setup: partial flag varieties

- G algebraic group$/\mathbb{C}$, $T \subset B \subset G$, $W = N(T)/T$,
- For $B \subset P$ a parabolic, $(G/P)^T \cong W_P \backslash W \cong W/W_P$.

For G of type A, B, C, D and P maximal, G/P is a **Grassmannian**.

Schubert classes

Schubert classes For $\pi \in W_P\backslash W$, the corresp. **Schubert class** is

$$S_\pi := \left[\overline{B^-\pi^{-1}P/P}\right] \in H_T^*(G/P).$$

Then $\{S_\pi\}_{\pi\in W_P\backslash W}$ freely generate $H_T^*(G/P)$ as an $H_T^*(\mathrm{pt})$–module.

Classical question: Determine the structure constants,

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Note: if $G/P = Gr(k,n)$, then (in H^*, not H_T^*) the $c_{\lambda\mu}^\nu$ are the *Littlewood-Richardson coefficients* for GL_k : $V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c_{\lambda\mu}^\nu}$

E.g. In $Gr(2,4)$, $(H_T^*(pt) \cong \mathbb{Z}[y_1, y_2, y_3, y_4])$:

$$S_{\square} \cdot S_{\square} = S_{\square\square} + S_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + (y_2 - y_3) S_{\square} \qquad (\text{in } H_T^*)$$

Grassmannians II

Involution: $\sigma : GL_{2n} \to GL_{2n}$, $X \mapsto J^{-1}(X^{-1})^{\mathrm{tr}} J$,
$J = \mathrm{Antidiag}(-1, \ldots, -1, 1, \ldots, 1)$.

$Sp_{2n} = GL_{2n}^{\sigma}$, $P = P_{GL_{2n}}$ parabolic of type $(k, 2n-k)$, $(k < n)$.

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Consider the *involution* $\lambda \mapsto \overline{\lambda}$ reversing λ and switching $0 \leftrightarrow 1$.
For $\widetilde{\iota}(\nu) := (\nu\overline{\nu}$ with 10's turned into 1's).

$$\begin{array}{ccc}
\{\nu \in (10)^{n-k}\{0,1\}^k\} \cong (SpGr(k,2n))^{T^n} & \xhookrightarrow{f_2} & SpGr(k,2n) \\
\Big\downarrow \widetilde{\iota} & & \Big\downarrow \iota \\
\{\lambda \in 0^k 1^{2n-k}\} \cong (Gr(k,2n))^{T^n} & \xhookrightarrow{f_1} & Gr(k,2n)
\end{array}$$

Note: We interchangeably consider binary strings $\pi \in 0^k 1^{2n-k}$ (i.e. in $W_P \backslash W$) and $\pi^{-1} \in W/W_P$.

Cohomology Rings

In equivariant cohomology, we get:

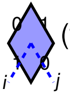
$$\begin{array}{ccc} H^*_{T^n}(SpGr(k,2n)^{T^n}) & \xleftarrow[f_2^*]{} & H^*_{T^n}(SpGr(k,2n)) \\ \uparrow (\tilde{\iota})^* & & \uparrow \iota^* \\ H^*_{T^n}(Gr(k,2n)^{T^n}) & \xleftarrow[f_1^*]{} & H^*_{T^n}(Gr(k,2n)) \end{array}$$

and since each f_i^* is injective (Kirwan), to understand ι^* we can instead compute in the left column.

Grassmannian (type A) Puzzles

A **puzzle** of size $2n$, 

, for $\lambda, \mu, \nu \in 0^k 1^{2n-k}$ is a tiling by the puzzle pieces:

, their rotations, and 

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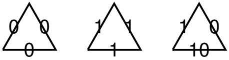
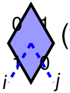


, their rotations, and 

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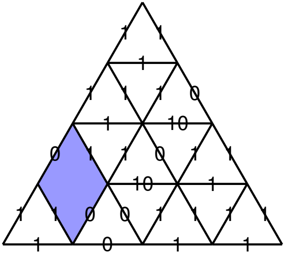
Example:



Schubert calculus

Theorem (Knutson-Tao '03, many extensions since)

For $\lambda, \mu \in 0^k 1^{2n-k}$, the product of S_λ and S_μ in $H_T^*(Gr(k,2n))$ is given by

$$S_\lambda \cdot S_\mu = \sum_{\nu \in 0^k 1^{2n-k}} \left(\sum_{\mathbf{P}} \left\{ v(\mathbf{P}) \; : \; \textit{puzzles } \mathbf{P} \textit{ with boundary } \overset{\lambda \;\; \mu}{\underset{\nu}{\triangle}} \right\} \right) S_\nu$$

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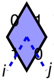
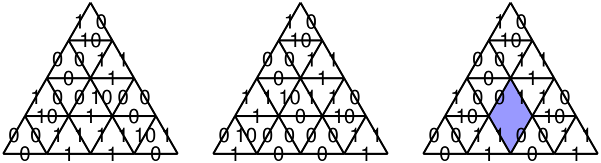
$$\left. \left. \right\} \right) S_\nu$$

where $v(\mathbf{P}) = \prod_{p \in \mathbf{P}} v(p)$, and for the individual pieces

- $v($  $) = 1$,
- $v($  $) = y_i - y_j \in \mathbb{Z}[y_1, \ldots, y_{2n}] \cong H_T^*(pt)$, if the rhombus sides face the i-th and j-th positions at the bottom of the puzzle.

Grassmann duality I

Example: $S_{0101} \cdot S_{0101} = S_{0110} + S_{1001} + (y_2 - y_3) S_{0101}$



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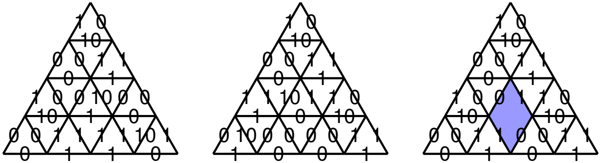
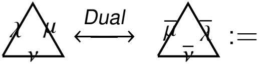


Grassmann duality

There is a ring isomorphism (from a homeom. of Grassmannians):

$$H_T^*(Gr(k, 2n)) \cong H_T^*(Gr(2n-k, 2n)), \quad S_\lambda \mapsto S_{\overline{\lambda}}$$

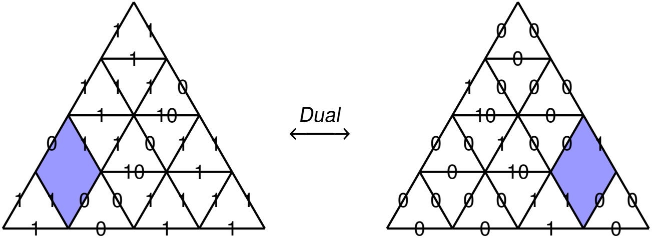
$$S_\lambda \cdot S_\mu \leftrightarrow S_{\overline{\mu}} \cdot S_{\overline{\lambda}}$$



:= reflect though vertical axis and swap 0 and 1

Grassmann duality II

For instance,



Question: What do self-dual puzzles count?

Branching from A to B/C

Let P, P' be the maximal parabolics of type $(k, 2n-k)$ in GL_{2n} and $(k, 2n+1-k)$ in GL_{2n+1} resp. Consider the Grassmannians:

$$Sp_{2n}/(Sp_{2n} \cap P_{k,2n}) = SpGr(k, 2n) \xhookrightarrow{\iota_{Sp}} Gr(k, 2n) = GL_{2n}/P$$

$$O_{2n+1}/(O_{2n+1} \cap P') = OGr(k, 2n+1) \xhookrightarrow{\iota_{O}} Gr(k, 2n+1) = GL_{2n+1}/P'$$

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$$O_{2n+1}/(O_{2n+1} \cap P') = OGr(k,2n+1) \overset{\iota_{O}}{\hookrightarrow} Gr(k,2n+1) = GL_{2n+1}/P'$$

$$\Rightarrow \quad \begin{array}{c} H_T^*(Gr(k,2n)) \xrightarrow{\iota_{Sp}^*} H_T^*(SpGr(k,2n)) \\ H_T^*(Gr(k,2n+1)) \xrightarrow{\iota_{O}^*} H_T^*(OGr(k,2n+1)) \end{array}$$

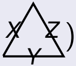
Main question: $\boxed{\iota_{Sp/O}^*(S_\lambda) = \sum_\nu c_\nu^\lambda S_\nu}$ $\quad c_\nu^\lambda = ??$

- Pragacz '00: (building on work of Stembridge) positive tableau formulæ for $H^*(Gr(n,2n)) \to H^*(SpGr(n,2n))$
- Coşkun '11: positive geometric rule for $H^*(Gr(k,2n))$

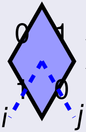
A combinatorial rule

Theorem (H–Knutson–Zinn-Justin '18)

$$\iota^*_{Sp/O}(S_\lambda) = \sum_{\nu \in W/W_P} \left(\sum_{\mathbf{P} \in \text{[puzzle]}} \prod_{p \in \mathbf{P}} v(p) \right) S_\nu$$

where $v(p) \in H^*_T(pt) = \mathbb{Z}[y_1, \ldots, y_n]$ *is given by* $v($ 

$) = 1$,

- $v($ 

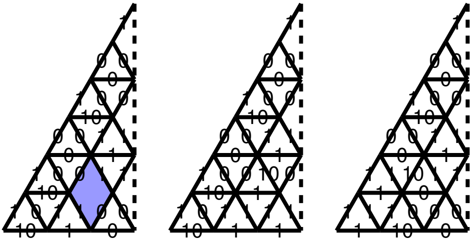
$) = \begin{cases} y_i - y_j, & j \leq n \\ y_i + y_{2n+1-j}, & n < j \end{cases}$
- $v($ 

$) = \begin{cases} 2, & G = Sp_{2n},\ (X, Y) = (0, 1) \\ 1 & \textit{otherwise} \end{cases}$

Examples and Goals

Remark: The values of v are given by R– and K–matrices in the 5–vertex model in statistical mechanics.

Example: $\iota^*(S_{110101}) = (y_2 - y_3)S_{10,1,0} + S_{10,1,1} + S_{1,10,0}$



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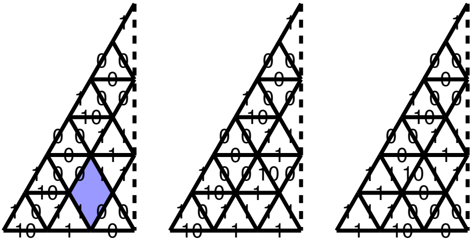


Goals: generalize to the 6–vertex model, understand the underlying geometry, obtain a generalized puzzle rule.

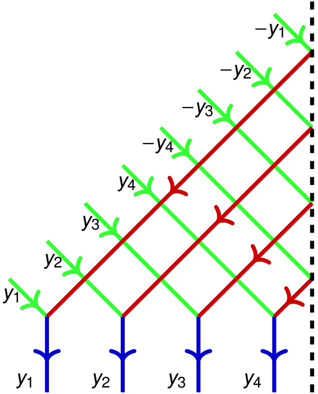
Tensor calculus

Idea of proof:

Consider the puzzle labels $\{0, 10, 1\}$ as indexing bases for $\mathbb{C}^3_G, \mathbb{C}^3_R, \mathbb{C}^3_B$.

We get a **scattering diagram** as the graph dual of a half-puzzle diagram, with assigned "spectral parameters"on the NW:

$$y_1, \ldots, y_n, -y_n, \ldots, -y_1.$$



Maps

Associate:

- to a crossing with parameters a and b, a linear map

$$R_{CD}(a-b) = \quad C \quad D$$

$$: \mathbb{C}^3_C \otimes \mathbb{C}^3_D \longrightarrow \mathbb{C}^3_D \otimes \mathbb{C}^3_C;$$

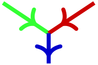
- to a wall-bounce of a strand with parameter a,

$$K_C(a) = \quad C$$

$$D \quad : \mathbb{C}^3_C \to \mathbb{C}^3_D, \quad (\text{and } a \mapsto -a);$$

- to a trivalent vertex with both parameters a,

$$U(a) =$$

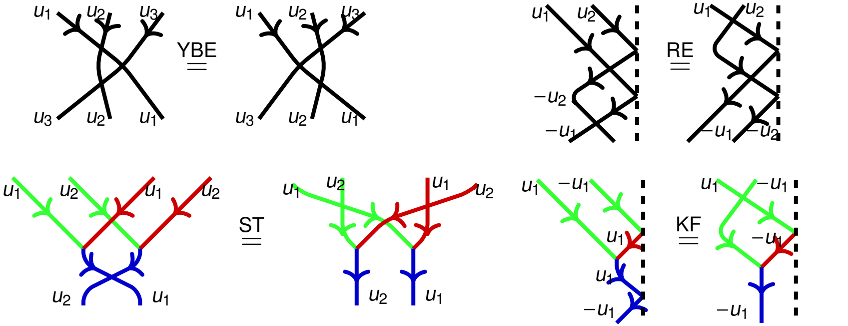


$$: \mathbb{C}^3_G \otimes \mathbb{C}^3_R \longrightarrow \mathbb{C}^3_B.$$

Gluing strands corresponds to composition, so the scattering diagram gives a linear map $\Phi : (\mathbb{C}^3_G)^{\otimes 2n} \longrightarrow (\mathbb{C}^3_B)^{\otimes n}$.

Relations

We ask that these maps satisfy the following identities:



E.g. $K_B(u_1) \circ U_{GR}(u_1) \circ (\mathrm{Id} \otimes K_G(-u_1)) \overset{\mathrm{KF}}{=} U_{GR}(-u_1) \circ (\mathrm{Id} \otimes K_G(u_1)) \circ R_{GG}(2u_1)$

The AJS/Billey formula ('94,'97)

Puzzle values: Let **P** range among all (self-dual) half-puzzles with labels , where $\lambda \in 0^k 1^{2n-k}$ and $\nu \in (10)^{n-k}\{0,1\}^k$. Then,

$$\text{the } (\nu, \lambda) \text{ matrix entry of } \Phi = \sum_{\mathbf{P}} v(\mathbf{P}) \quad (\text{Goal: ``} = c_\nu^\lambda\text{''}).$$

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Next, computing restriction to T-fixed points:

Proposition (AJS/Billey using scattering diagrams)

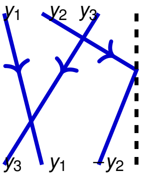
Let $\lambda, \mu \in W_P \backslash W$ (strings in $\{0, 10, 1\}$), where W is of type A or C, and P maximal. To compute $S_\lambda|_\mu$:

1. *Make a scattering diagram by taking a reduced word for the shortest lift $\widetilde{\mu}^{-1}$.*
2. *Replace crossings with R_{BB} and reflections with K_B.*

Then $S_\lambda|_\mu$ is the $(id_{G/P}, \lambda)$ matrix entry of the resulting map.

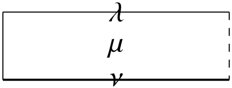
Example

For $G = Sp_6$ and $\widetilde{\mu}^{-1} = s_2 s_3 s_1$, the scattering diagram is



$$(R_{BB}(y_1 - y_3) \otimes \mathrm{Id}) \circ (Id^{\otimes 2} \otimes K_B(y_2)) \circ (\mathrm{Id} \otimes R_{BB}(y_2 - y_3)) : (\mathbb{C}_B^3)^{\otimes 3} \to (\mathbb{C}_B^3)^{\otimes 3}$$

For $\lambda, \mu, \nu \in W_P \backslash W$ as above, we denote

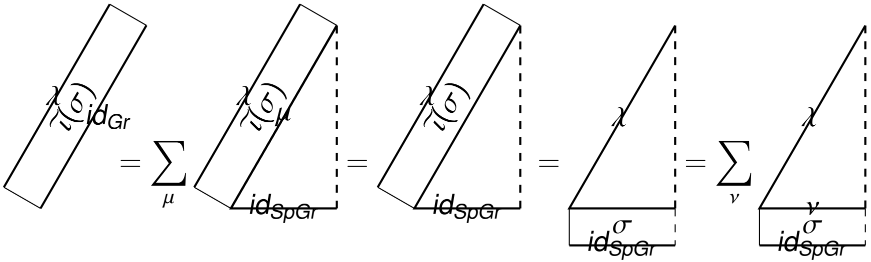


$:=$ the (ν, λ) matrix entry for the map coming from a reduced word for $\widetilde{\mu}^{-1}$.

By the proposition, when $\nu = id_{G/P}$ this gives $S_\lambda|_\mu$.

Theorem proof (sketch) I

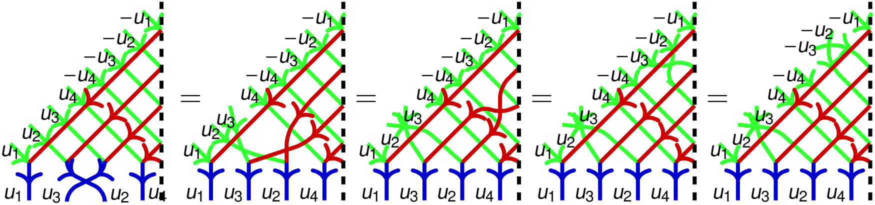
In $H_T^*(\mathrm{pt})$, we have the following equality

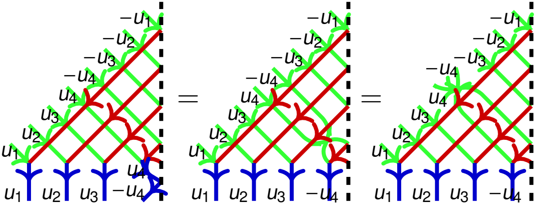


In the second and fourth equality, the strings μ and ν have content $0^k 1^{2n-k}$ and $(10)^{n-k}\{0,1\}^k$ respectively, and all other terms of the sum vanish.

Theorem proof (sketch) II

The third equality above follows from the following operations on scattering diagrams:





Lagrangian correspondences

A **Lagrangian correspondence** L between two symplectic manifolds A and B, $A \overset{L}{\longleftrightarrow} B$, is:

A Lagrangian cycle L in $(-A) \times B$
(equivalently L in $A \times (-B)$).

If $T \curvearrowright A, B$ and L is T–invariant, then

$$H_T^*(A) \xrightarrow{(\pi_A)^*} H_T^*(A \times B) \xrightarrow{\cup [L]} H_T^*(A \times B) \xrightarrow{(\pi_B)_*} H_T^*(B) \cong H_T^*(B)$$

Note: In our setting, will work with T^*G/P.

Examples

1. *Symplectic reduction*
   For $T \subseteq G \curvearrowright X$ Hamiltonian action, have a moment map $X \xrightarrow{\mu} \mathfrak{g}^*$. Take a regular point a for μ s.t. $a \in (\mathfrak{g}^*)^G$ Let $Z = \mu^{-1}(a),\ Y = \mu^{-1}(a)//G$. Then $X \hookleftarrow Z \twoheadrightarrow Y$.
   [Marsden-Weinstein '74] ∃! symplectic structure on Y s.t. $Z \subseteq (-X) \times Y$ is Lagrangian.

2. *Maulik–Okounkov stable envelopes*
   Suppose $S \curvearrowright X$ is a sympl. res. with a circle action.
   Let C be a fixed point component.
   The **stable envelope construction** produces a certain Lagrangian cycle $L = \overline{Attr(C)} + \ldots$ in $(-C) \times X$.

Maulik–Okounkov classes

For a regular circle action $S \curvearrowright T^*G/P$ and a fixed pt. $\lambda \in W/W_P$, the stable envelope construction produces an MO cycle

$$MO_\lambda = \overline{BB}_\lambda + \sum_{\mu \leq \lambda} a_{\lambda,\mu} \overline{BB}_\mu, \quad a_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$$

$BB_\lambda = Attr(\lambda) = CX^o_\lambda :=$ conormal bundle of the Bruhat cell X^o_λ.

This in turn gives a class $[MO_\lambda] \in H^*_{T\times\mathbb{C}^\times}(T^*G/P) \cong H^*_T(G/P)[\hbar]$.

Maulik–Okounkov classes

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Segre–Schwartz–MacPherson:

$$SSM_\lambda = \frac{[MO_\lambda]}{[\text{zero section}]} \in \widetilde{H}^0_{T\times\mathbb{C}^\times}(T^*G/P)$$

$$\Rightarrow SSM_\lambda = \hbar^{-\ell(\lambda)} S_\lambda + \text{l.o.t}(\hbar) \qquad \Rightarrow S_\lambda = \lim_{\hbar\to\infty} \left(SSM_\lambda \cdot \hbar^{\ell(\lambda)}\right)$$

Structure constants: $\quad c_{\lambda\mu}^{\nu} = \lim_{\hbar\to\infty} \left((c')_{\lambda\mu}^{\nu} \cdot \hbar^{\ell(\lambda)+\ell(\mu)-\ell(\nu)}\right)$

The Sp_{2n} case

Theorem in progress (H–Knutson–Zinn-Justin '20)

There are Lagrangian correspondences

$$\lambda \overset{L_1}{\longleftrightarrow} T^*Gr(k,2n) \overset{L_2}{\longleftrightarrow} T^*OGr(k,4n) \overset{L_3}{\longleftrightarrow} T^*SpGr(k,2n)$$

that compute the restriction of SSM classes, and together with the 6-vertex R- and K-matrices realize a puzzle rule.

- $L_1 = MO_\lambda$ is the stable envelope for the circle action
$$S_1 \cong Diag(t, t^2, \dots, t^{2n}).$$
- $L_2 = Attr(T^*Gr(k,2n))$ is the stable envelope for the circle
$$S_2 \cong Diag(t, \dots, t, t^{-1}, \dots, t^{-1}).$$
- L_3 is obtained by symplectic reduction.

Construction details 1/2

Consider the parabolic P given by $\mathfrak{o}(4n) \supseteq \mathfrak{p} = \mathfrak{l} \ltimes rad(\mathfrak{p})$, as below

$$\left\{X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid JX + X^{tr}J = 0\right\} \supseteq \left\{\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}\right\} = \left\{\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right\} \ltimes \left\{\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\right\}$$

where J is the form given by, for $J' = Antidiag(1, \ldots, 1, -1, \ldots, -1)$,

$$J = \begin{bmatrix} 0 & J' \\ (J')^{tr} & 0 \end{bmatrix}$$

$$O(4n) \curvearrowright T^*OGr(k, 4n) \xrightarrow{\phi} o(4n)^* \rightarrow rad(p)^* \cong o(4n)/p$$

$$(X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, V) \mapsto X \mapsto B$$

This gives a P–equivariant and $Rad(P)$–invariant map,

$$\mu : T^*OGr(k, 4n) \rightarrow o(4n)/p.$$

Construction details 2/2

The Levi $L \cong GL(2n)$ has a subgroup $Sp(2n)$ that preserves the fiber $\{B = \mathbb{1}\}$ of μ, and we get

$$Sp(2n) \curvearrowright \mu^{-1}(\mathbb{1})/Rad(P) \cong T^*SpGr(k,2n)$$

The isomorphism is obtained from:

$$\begin{array}{ccc} \mu^{-1}(\mathbb{1}) & \xrightarrow{f} & T^*SpGr(k,2n) \\ \downarrow \phi & & \downarrow \psi \\ \mathfrak{o}(4n)^* \supseteq \mathrm{Im}(\phi) & \xrightarrow{f'} & \mathrm{Im}(\psi) \subseteq \mathfrak{sp}(2n)^* \end{array}$$

$$f' : X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto A + D.$$

The End

Thank you!