GENERALIZED SLICES FOR MINUSCULE COCHARACTERS

VASILY KRYLOV

This talk is based on the joint work in progress with Ivan Perunov (see [KP19]).

1. Definitions and motivations

Let us start from some definitions. We fix a triple $G \supset B \supset T$, consisting of a connected reductive algebraic group over $\mathbb{C}$, a Borel subgroup B and a maximal torus T; $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{t}$ are their Lie algebras. We denote by $B_- \supset T$ the opposite Borel subgroup of G. We denote by Λ the coweight lattice of (T, G) and by $\Lambda^+ \subset \Lambda$ the submonoid of dominant coweights. We denote by $\Delta^\vee$ (resp. Δ) the set of roots (resp. coroots) of (T, G) and by $\Delta_+^\vee$ (resp. Δ_+) the set of positive roots (resp. coroots) with respect to the Borel subgroup $B \subset G$. We also denote by $I \subset \Delta_+$ (resp. $I^\vee \subset \Delta_+^\vee$) the set of simple coroots (resp. roots). We will denote by W the Weyl group of (T, G) and by $w_0 \in W$ the longest element of W (with respect to B).

1.1. Affine Grassmannian.

Definition 1.1 *(Affine Grassmannian)*
Let Gr_G *be the moduli space of* G*-bundles* $\mathcal{P}$ *over* $\mathbb{P}^1$ *with a trivialization* σ *outside* 0, $\sigma\colon \mathcal{P}^{triv}|_{\mathbb{P}^1\setminus\{0\}} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{P}^1\setminus\{0\}}$.

Set $\mathcal{K} := \mathbb{C}((z))$, $\mathcal{O} := \mathbb{C}[[z]]$ and denote $D := \operatorname{Spec}\mathcal{O}$, $\mathring{D} := \operatorname{Spec}\mathcal{K}$. One can think about D as about a formal disk around a point $0 \in \mathbb{P}^1$ and $\mathring{D} \subset D$ is a punctured (formal) neighbourhood of $0 \in \mathbb{P}^1$. The following proposition was proved by Beauville-Laszlo:

Proposition 1.2
Gr_G *coincides with the moduli space of* G*-bundles* $\mathcal{P}_D$ *on* D *together with a trivialization* $\sigma_{\mathring{D}}$ *of* $\mathcal{P}_D$ *being restricted to* $\mathring{D}$.

Proof. It is clear that a point $(\mathcal{P}, \sigma) \in \mathrm{Gr}_G$ defines us a vector bundle $\mathcal{P}_D$ with a trivialization $\sigma_{\mathring{D}}$ as follows: $\mathcal{P}_D := \mathcal{P}|_D$, $\sigma_{\mathring{D}} := \sigma|_{\mathring{D}}$.

In the opposite direction if we have a pair $(\mathcal{P}_D, \sigma_{\mathring{D}})$ we can then glue vector bundles $\mathcal{P}_D$ and $\mathcal{P}^{triv}|_{\mathbb{P}^1\setminus\{0\}}$ using the trivialization $\sigma_{\mathring{D}}$ and obtain the desired bundle $\mathcal{P}$. To obtain the trivialization σ it remains to note that by the construction $\mathcal{P}|_{\mathbb{P}^1\setminus\{0\}} = \mathcal{P}^{triv}|_{\mathbb{P}^1\setminus\{0\}}$. □

It follows from Proposition 1.2 that the space Gr_G is just the quotient $G(\mathcal{K})/G(\mathcal{O})$. Any cocharacter $\lambda \in \Lambda$ defines us an element of $\mathrm{Map}(\mathbb{C}^\times, G) = G(\mathbb{C}[z, z^{-1}])$ so gives rise to an element of Gr_G to be denoted by z^λ. The group $G(\mathcal{O})$ acts on Gr_G via left multiplication.

For $\lambda \in \Lambda^+$, denote by Gr_G^λ the $G(\mathcal{O})$-orbit of z^λ. We have the following decompositions:

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}_G^\lambda, \ \overline{\mathrm{Gr}}_G^\lambda = \bigsqcup_{\mu \leqslant \lambda} \mathrm{Gr}_G^\mu, \tag{1.1}$$

where $\leqslant$ is the standard dominance order ($\mu \leqslant \lambda$ if their difference is the linear combination of positive coroots).

Remark 1.3
One can think about this decomposition as about the (parabolic) Bruhat decomposition corresponding to a parabolic subgroup $G(\mathcal{O}) \subset G(\mathcal{K})$.

For any $\lambda \in \Lambda^+$ space $\overline{\mathrm{Gr}}_G^\lambda$ is a projective algebraic variety of dimension $\langle 2\rho^\vee, \lambda \rangle$ and $\mathrm{Gr}_G^\lambda \subset \overline{\mathrm{Gr}}_G^\lambda$ is an open smooth subvariety, here $2\rho^\vee$ is the sum of positive roots. We see that $\mathrm{Gr}_G = \varinjlim \overline{\mathrm{Gr}}_G^\lambda$ is an ind-projective scheme.

We have an action $\mathbb{C}^\times \curvearrowright \mathrm{Gr}_G$ via *loop rotation*:

$$t \cdot g := (z \mapsto g(tz)),\ t \in \mathbb{C}^\times,\ g(z) \in G(\mathcal{K}).$$

The fixed points of $\mathbb{C}^\times \curvearrowright \mathrm{Gr}_G$ are $\bigsqcup_{\mu \in \Lambda^+} Gz^\mu$.

It is easy to see that

$$\mathrm{Gr}_G^\mu = \{x \in \mathrm{Gr}_G \mid \lim_{t \to 0} t \cdot x \in Gz^\mu\}$$

so in other words Gr_G^μ is *attractor* to Gz^μ w.r.t. the loop rotation.

1.2. **Transversal slices in** Gr_G**.** Variety Gr_G has another decomposition ("opposite" to (1.1)) corresponding to $G[z^{-1}]$-orbits.

For any $\mu \in \Lambda^+$ set $\mathrm{Gr}_{G,\mu} := G[z^{-1}] \cdot z^\mu$. Then it follows from the Grothendieck-Birkhoff theorem (see also [Ram83]) about classification of G-bundles on $\mathbb{P}^1$ and using that $G[z^{-1}] \backslash G(\mathcal{K})/G(\mathcal{O}) = \mathrm{Bun}_G(\mathbb{P}^1)$ we obtain

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}_{G,\lambda}\,.$$

Directly by the definitions

$$\mathrm{Gr}_{G,\lambda} = \{x \in \mathrm{Gr}_G \mid \lim_{t \to \infty} t \cdot x \in Gz^\lambda\},\ \mathrm{Gr}_{G,\mu} \cap \mathrm{Gr}_G^\mu = Gz^\mu.$$

Let us denote by $G[z^{-1}]_1 \subset G[z^{-1}]$ the kernel of the natural evaluation at infinity homomorphism $G[z^{-1}] \to G$.

We set $\mathcal{W}_\mu := G[z^{-1}]_1 \cdot z^\mu$. By the definitions

$$\mathcal{W}_\mu = \{x \in \mathrm{Gr}_G \mid \lim_{t \to \infty} t \cdot x = z^\mu\},\ \mathcal{W}_\mu \cap \mathrm{Gr}_G^\mu = \{z^\mu\}.$$

We can now finally define transversal slices as follows.

Definition 1.4 *(Transversal slices)*
For $\lambda \geqslant \mu$ (otherwise $\overline{\mathcal{W}}_\mu^\lambda$ will be empty), $\lambda, \mu \in \Lambda^+$ we set

$$\mathcal{W}_\mu^\lambda := \mathrm{Gr}_G^\lambda \cap \mathcal{W}_\mu,\ \overline{\mathcal{W}}_\mu^\lambda := \overline{\mathrm{Gr}}_G^\lambda \cap \mathcal{W}_\mu.$$

Example 1.5. *For $\mu = \lambda$ we have $\overline{\mathcal{W}}_\lambda^\lambda = \{z^\lambda\}$.*

Remark 1.6
Note that in the definition it is not necessary to assume that $\mu \in \Lambda^+$. If we take some μ and define $\overline{\mathcal{W}}^\lambda_\mu$ as above then it will be isomorphic to the variety $\overline{\mathcal{W}}^\lambda_{\mu^+}$, where $\mu^+ \in W\mu$ is the dominant representative of μ. Indeed the isomorphism is given by the conjugation by an element $w \in W$ such that $w\mu = \mu^+$.

Variety $\overline{\mathcal{W}}^\lambda_\mu$ is an affine variety of dimension $\langle 2\rho^\vee, \lambda - \mu\rangle$ equipped with a contracting action of $\mathbb{C}^\times$, $\mathcal{W}^\lambda_\mu \subset \overline{\mathcal{W}}^\lambda_\mu$ is an open smooth subvariety.

Remark 1.7
$\overline{\mathcal{W}}^\lambda_\mu$ is affine since it is a closed subvariety of an (ind-)affine scheme $\mathcal{W}_\mu$.

Remark 1.8
Variety $\overline{\mathcal{W}}^\lambda_\mu$ is a transversal slice to Gr^μ_G inside $\overline{\mathrm{Gr}}^\lambda_G$ at the point z^μ in the following sence: there exists an open subset $U \subset \mathrm{Gr}^\mu_G$ and an open embedding $U \times \overline{\mathcal{W}}^\lambda_\mu \hookrightarrow \overline{\mathrm{Gr}}^\lambda_G$ such that the following diagram is commutative:

$$\begin{array}{ccc} U \times \{z^\mu\} & \longrightarrow & \mathrm{Gr}^\mu_G \times \{z^\mu\} \\ \downarrow & & \downarrow \\ U \times \overline{\mathcal{W}}^\lambda_\mu & \longrightarrow & \overline{\mathrm{Gr}}^\lambda_G. \end{array}$$

1.3. **Moduli definition of slices and generalization.** Let us now try to find a moduli definition of $\overline{\mathcal{W}}^\lambda_\mu$ (in the spirit of the definition of Gr_G), we follow [BF14, Proof of Theorem 2.8]. For simplicity we assume that μ is *regular* (i.e. $\langle \alpha^\vee, \mu\rangle \neq 0$, $\forall \alpha^\vee \in \Delta^\vee$). In this case we have an isomorphism $G \cdot z^\mu \simeq G/B_-$ (since the stabilizer of z^μ in G is B_-).

By the definition we have

$$\overline{\mathcal{W}}^\lambda_\mu = \overline{\mathrm{Gr}}^\lambda_G \cap \mathcal{W}_\mu$$

so to find the moduli description of $\overline{\mathcal{W}}^\lambda_\mu$ it is enough to find moduli descriptions of $\overline{\mathrm{Gr}}^\lambda_G$, $\mathcal{W}_\mu$.

First of all recall that $\overline{\mathrm{Gr}}^\lambda_G$ parametrizes pairs $(\mathcal{P}, \sigma)$ such that σ has pole of order $\leqslant \lambda$ at the point 0.

Remark 1.9
Trivialization σ of $\mathcal{P}$ out of 0 has pole $\leqslant \lambda$ at 0 if for any irreducible highest weight representation $V^{\eta^\vee}$ of G the pole (maximum of poles of matrix elements) of the induced trivialization $\sigma_{V^{\eta^\vee}}$ of $\mathcal{P} \times_G V^{\eta^\vee}$ at 0 is $\leqslant \langle \eta^\vee, \lambda\rangle$.

Let us now deal with $\mathcal{W}_\mu = G[z^{-1}]_1 \cdot z^\mu \subset G[z^{-1}] \cdot z^\mu$. Note that $G[z^{-1}] \cdot z^\mu \subset \mathrm{Gr}_G$ parametrizes pairs $(\mathcal{P}, \sigma)$ as above such that $\mathcal{P}$ has type μ (i.e. $\mathcal{P} \simeq \mathcal{O}(\mu)$), here $\mathcal{O}(\mu)$ is a G-bundle corresponding to a point $z^\mu \in \mathrm{Gr}_G$ (for $G = GL_N$, $\mu = (a_1, \ldots, a_N)$ we have $\mathcal{O}(\mu) = \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_N)$).

Let us now give the moduli description of a subspace $\mathcal{W}_\mu = G[z^{-1}]_1 \cdot z^\mu \subset G[z^{-1}] \cdot z^\mu$. Recall that we have an evaluation at infinity morphism

$$\mathrm{ev}_\infty \colon G[z^{-1}] \cdot z^\mu \twoheadrightarrow G \cdot z^\mu \simeq G/B_-$$

and $\mathcal{W}_\mu$ is the fiber $\mathrm{ev}_\infty^{-1}(z^\mu)$. Recall also that μ is *regular* and *dominant* so any vector bundle $\mathcal{P}$ of type μ has a *canonical* B_--structure $\mathcal{P}^{B_-} \subset \mathcal{P}$ of degree μ that comes from a Harder-Narasimhan filtration. We conclude from the definitions that $\mathcal{W}_\mu$ is a moduli space of pairs $(\mathcal{P}, \sigma)$ such that $\mathcal{P}$ has type μ and moreover the fiber of the Harder-Narasimhan B_--structure $\mathcal{P}^{B_-}$ at ∞ coincides with the fiber of the Harder-Narasimhan B_--structure of z^μ at ∞ so equals to B (w.r.t. σ).

Remark 1.10
Recall that to any B-bundle $\mathcal{E}$ we can associate its degree $\deg \mathcal{E} \in \Lambda$ which is the degree of the T-bundle $\mathcal{E} \times_B T$, where B acts on T via the natural surjection $B \twoheadrightarrow T$.

Recall again that μ is *dominant* and *regular* so if some vector bundle $\mathcal{P}$ has a B_--structure of degree μ then $\mathcal{P}$ *must* have type μ and the B_--structure in $\mathcal{P}$ of degree μ is unique (think about the *embeddings of vector bundles* $\mathcal{O}(m) \hookrightarrow \mathcal{O}(n) \oplus \mathcal{O}(-n)$ with $m, n \geqslant 0$).

It follows that $\mathcal{W}_\mu$ is a moduli space of triples $(\mathcal{P}, \sigma, \mathcal{P}^{B_-})$ such that $\mathcal{P}$, σ are as above and $\mathcal{P}^{B_-}$ is a B_--structure in $\mathcal{P}$ such that the fiber of $\mathcal{P}^{B_-}$ at ∞ is B (w.r.t. σ).

So we come to the following (equivalent) moduli definition of transversal slices (we follow the notations of [BFN16, Section 2(ii)]):

Definition 1.11 *(Generalized transversal slices)*
$\overline{\mathcal{W}}^\lambda_\mu$ is the moduli space of triples $(\mathcal{P}, \sigma, \phi)$, where $\mathcal{P}$ is a G-bundle on $\mathbb{P}^1$, σ is a trivialization of $\mathcal{P}$ outside 0 with pole at 0 of order $\leqslant \lambda$ and ϕ is a B-structure in $\mathcal{P}$ of degree $w_0\mu$ and such that $\phi|_\infty = B_-$.

Remark 1.12
One technical detail – we have seen that in the definition of $\overline{\mathcal{W}}^\lambda_\mu$ one gets naturally B_--structures in $\mathcal{P}$ of degree μ. If we want to pass to B-structures then the degree will be $w_0\mu$ since $B = w_0 B_- w_0$.

Remark 1.13
One can rewrite this definition as follows:

$$\overline{\mathcal{W}}^\lambda_\mu = \overline{\mathrm{Gr}}^\lambda_G \times_{'\mathrm{Bun}_G} \mathrm{Bun}_B^{w_0\mu},$$

where $'\mathrm{Bun}_G$ is the moduli space of G-bundles on $\mathbb{P}^1$ together with a trivialization at ∞ and $\mathrm{Bun}_B^{w_0\mu}$ is the moduli space of B-bundles on $\mathbb{P}^1$ of degree $w_0\mu$.

Note now that for this definition μ does not need to be dominant! Varieties $\overline{\mathcal{W}}^\lambda_\mu$ for nondominant are called *generalized transversal slices*.

We still have a natural morphism $\mathbf{p} \colon \overline{\mathcal{W}}^\lambda_\mu \to \overline{\mathrm{Gr}}^\lambda_G$ but it is not an embedding in general. This happens because for nondominat μ there can be many different B-structures in a fixed G-bundle $\mathcal{P}$. Variety $\overline{\mathcal{W}}^\lambda_\mu$ is still affine of dimension $2\langle \rho^\vee, \lambda - \mu \rangle$.

Example 1.14. *Consider the case "opposite" to the case when μ is dominant: $\lambda = 0$, $\mu \leqslant 0$. Then the order of pole of σ at 0 is $\leqslant 0$, hence, σ extends to a trivialization of $\mathcal{P}$. We see that $\overline{\mathcal{W}}^0_\mu$ is the moduli space of B-structures of degree $w_0\mu$ in the trivial bundle $\mathcal{P}^{triv}$. Note that a B-structure in $\mathcal{P}^{triv}$ is the same as a section of $\mathcal{P}^{triv}/B$ i.e. a morphism $\mathbb{P}^1 \to G/B$ of degree $w_0\mu$. We obtain the so-called open Zastava space.*

For $G = \mathrm{SL}_2$ we have $G/B \simeq \mathbb{P}^1$ and μ just corresponds to some number $-n \in \mathbb{Z}_{\leqslant 0}$ so $w_0\mu = n \in \mathbb{Z}_{\geqslant 0}$ so we are considering maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree n and this is a complement of a hypersurface in $\mathbb{P}^{2n}$.

Remark 1.15
In general we have a locally closed embedding

$$\iota\colon \overline{\mathcal{W}}^\lambda_\mu \hookrightarrow \overline{\mathrm{Gr}}^\lambda_G \times Z^{-w_0(\lambda-\mu)}.$$

1.4. Relation of slices $\overline{\mathcal{W}}^\lambda_\mu$ to the representation theory.

Definition 1.16 *(Action of T)*
T acts on $\overline{\mathcal{W}}^\lambda_\mu$, Gr_G via changing the trivialization.

Proposition 1.17
The set of T-fixed points $(\overline{\mathcal{W}}^\lambda_\mu)^T$ consists of one element if μ is a weight of V^λ (the irreducible representation of the Langlands dual group $G^\vee$ with the highest weight λ) and is empty otherwise. We denote the corresponding fixed point by z^μ.

Proof. Follows from [Kry18, Lemma 2.8]. □

Definition 1.18 *(Repellents and Attractors)*
The repellent (resp. attractor) to the (unique) T-fixed point $z^\mu \in \overline{\mathcal{W}}^\lambda_\mu$ is defined as

$\mathcal{R}^\lambda_\mu := \{x \in \overline{\mathcal{W}}^\lambda_\mu | \lim_{t\to\infty} 2\rho(t)\cdot x = z^\mu\}$ (resp. $\mathcal{A}^\lambda_\mu := \{x \in \overline{\mathcal{W}}^\lambda_\mu | \lim_{t\to 0} 2\rho(t)\cdot x = z^\mu\}$), *where $2\rho \in \Lambda$ is the sum of positive coroots.*

The following theorem (see [Kry18], can be deduced from the results of Braverman-Gaitsgory) relates varieties $\overline{\mathcal{W}}^\lambda_\mu$ with representation theory.

Theorem 1.19
The set $\mathbf{B}(\lambda) := \bigsqcup_{\mu\in\Lambda} \mathrm{Irr}_{top}(\mathcal{R}^\lambda_\mu)$ has a crystal structure over $G^\vee$ of a highest weight λ, here $\mathrm{Irr}_{top}(\mathcal{R}^\lambda_\mu)$ is the set of top-dimensional irreducible components of $\mathcal{R}^\lambda_\mu$. Moreover $\mathrm{Irr}_{top}(\mathcal{R}^\lambda_\mu)$ is the subset of crystal $\mathbf{B}(\lambda)$ of weight μ.

Remark 1.20
Note that it is crucial in Theorem 1.19 that μ may be nondominant.

Corollary 1.21
For any $\mu \in \Lambda$ the dimension of the μ-weight space V^λ_μ coincides with the number of top dimensional irreducible components of $\mathcal{R}^\lambda_\mu$.

Remark 1.22
Recall that A $G^\vee$-crystal is a set $\mathbf{B}$ together with maps:

(1) $wt : \mathbf{B} \to \Lambda,\ \varepsilon_i,\ \varphi_i : \mathbf{B} \to \mathbb{Z} \cup \infty$.

(2) $\mathrm{e}_i,\ \mathrm{f}_i : \mathbf{B} \to \mathbf{B} \cup \{0\}$.

such that for each $i \in I$ we have a)–c):

(a) For each $\mathbf{b} \in \mathbf{B}$, $\varphi_i(\mathbf{b}) = \varepsilon_i(\mathbf{b}) + \langle wt(\mathbf{b}), \alpha_i^\vee \rangle$.

(b) Let $\mathbf{b} \in \mathbf{B}$. *If* $\mathrm{e}_i \cdot \mathbf{b} \in \mathbf{B}$ *for some i, then*

$$wt(\mathrm{e}_i \cdot \mathbf{b}) = wt(\mathbf{b}) + \alpha_i,\ \varepsilon_i(\mathrm{e}_i \cdot \mathbf{b}) = \varepsilon_i(\mathbf{b}) - 1,\ \varphi_i(\mathrm{e}_i \cdot \mathbf{b}) = \varphi_i(\mathbf{b}) + 1.$$

If $\mathrm{f}_i \cdot \mathbf{b} \in \mathbf{B}$ *for some i, then*

$$wt(\mathrm{f}_i \cdot \mathbf{b}) = wt(\mathbf{b}) - \alpha_i,\ \varepsilon_i(\mathrm{f}_i \cdot \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1,\ \varphi_i(\mathrm{f}_i \cdot \mathbf{b}) = \varphi_i(\mathbf{b}) - 1.$$

(c) For all $\mathbf{b}, \hat{\mathbf{b}} \in \mathbf{B}$, $\mathrm{e}_i \cdot \mathbf{b} = \hat{\mathbf{b}}$ *if and only if* $\mathbf{b} = \mathrm{f}_i \cdot \hat{\mathbf{b}}$.

Remark 1.23
Actually one has more then just a crystal structure on every $\mathbf{B}(\lambda)$. One can also prove that projection morphisms $\mathbf{B}(\lambda_1) \otimes \mathbf{B}(\lambda_2) \twoheadrightarrow \mathbf{B}(\lambda_1 + \lambda_2) \cup \{0\}$ are induced by the natural multiplication morphisms $\overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times \overline{\mathcal{W}}^{\lambda_2}_{\mu_2} \to \overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}$ which we do not discuss in this talk.

1.5. **Some motivation.** Set $\lambda^* := -w_0\lambda$, $\mu^* := -w_0\mu$. In paper [BFN16] authors have proved that in types ADE (there is a generalization of this fact to other types, see [NW19]) varieties $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$ are isomorphic to (framed) Coulomb branches corresponding to a quiver Q (of type ADE) and dimension vectors $\underline{v}$ and framing $\underline{w}$ such that

$$\lambda = \sum_i w_i\omega_i,\ \mu = \sum_i w_i\omega_i - \sum_i v_i\alpha_i,$$

where ω_i are fundamental coweights of G and α_i are simple coroots.

In the same paper authors conjectured that symplectically dual varieties to Coulomb branches are quiver varieties for Q, dimension vector $\underline{v}$ and framing $\underline{w}$.

From the representation-theoretic point of view the simplest dominant $\lambda \in \Lambda^+$ are so-called *minuscule* coweights. Let us recall the definition.

Definition 1.24 *(Minuscule coweights)*
A coweight $\lambda \in \Lambda^+$ is called minuscule if for any coweight $\mu \in \Lambda$, such that $V^\lambda_\mu \neq \{0\}$ we have $\mu \in W\lambda$. Here V^λ is the irreducible representation of the Langlands dual group $G^\vee$ with the highest weight λ and W is the Weyl group of $G^\vee$.

Example 1.25. *In type A any fundamental coweight is minuscule (since the corresponding representations are wedge powers of the standard representation). In general any minuscule coweight is fundamental but the opposite implication is wrong, the set of minuscule coweights is a subset of the set of fundamental coweights.*

It is clear from (1.1) that for a minuscule λ we have $\overline{\mathrm{Gr}}^\lambda_G = \mathrm{Gr}^\lambda_G$, hence, $\overline{\mathcal{W}}^\lambda_\mu = \mathcal{W}^\lambda_\mu$ so $\overline{\mathcal{W}}^\lambda_\mu$ is actually *smooth* and symplectic (symplectic form comes from its description as a Coulomb branch).

At the symplectic dual side we obtain quiver varieties $\mathfrak{M}(\underline{v},\underline{w})$. Recall that

$$\lambda=\sum_i w_i\omega_i$$

and recall also that λ must be fundamental (since it is minuscule). It follows that $\lambda=\omega_k$ for some k, hence, $\underline{w}=(0,0,\dots,0,1,0,0,\dots,0)$ with 1 standing on k-th place.

Lemma 1.26
Let λ be minuscule and $\lambda\geqslant\mu$ be a weight of V^λ (i.e. $\mu\in W\lambda$). Then if $\underline{v}$, $\underline{w}$ are such that

$$\lambda=\sum_i w_i\omega_i,\ \mu=\lambda-\sum_i v_i\alpha_i$$

then the Nakajima quiver variety $\mathfrak{M}(\underline{v},\underline{d})$ consists of one point:

$$\mathfrak{M}(\underline{v},\underline{w})=\mathrm{pt}\,.$$

Proof. Recall that the dimension of $\mathfrak{M}(\underline{v},\underline{d})$ equals to (C is the Cartan matrix for $\mathfrak{g}$)

$$\underline{v}\cdot(2\underline{w}-C\underline{v})=2(\sum_i v_i\alpha_i,\sum_j w_j\omega_j)-(\sum_i v_i\alpha_i,\sum_j v_j\alpha_j)=$$
$$=2(\lambda,\lambda-\mu)-(\lambda-\mu,\lambda-\mu)=(\lambda,\lambda)-(\mu,\mu).$$

It remains to recall that $\mu\in W\lambda$ so

$$\dim\mathfrak{M}(\underline{v},\underline{w})=(\lambda,\lambda)-(\mu,\mu)=0.$$

It now follows from the fact that μ is a weight of V^λ that $\mathfrak{M}(\underline{v},\underline{w})\neq\varnothing$. The variety $\mathfrak{M}(\underline{v},\underline{w})$ is connected so we conclude that $\mathfrak{M}(\underline{v},\underline{w})$ consists of one point. □

We see that symplectically dual varieties to (generalized) transversal slices $\overline{\mathcal{W}}^\lambda_\mu$ with $\mu\in W\lambda$ and λ minuscule are points. Recall also that $\dim\overline{\mathcal{W}}^\lambda_\mu=2\langle\rho^\vee,\lambda-\mu\rangle$. It is natural to conjecture then that $\overline{\mathcal{W}}^\lambda_\mu$ is "as simple as possible":

Conjecture 1.27
With the assumptions as above we have $\overline{\mathcal{W}}^\lambda_\mu\simeq\mathbb{A}^{2\langle\rho^\vee,\lambda-\mu\rangle}$.

Remark 1.28
The proof of this conjecture was known to Nakajima in type A. The proof uses the identification of $\overline{\mathcal{W}}^\lambda_\mu$ with so-called bow varieties introduced by Nakajima and Takayama in the paper [NT17]*. Such an identification exists only in type A.*

2. Main results

Theorem 2.29
For a minuscule $\lambda\in\Lambda^+$ and $\mu\in W\lambda$, we have

$$\mathcal{R}^\lambda_\mu\simeq\mathbb{A}^{\langle\rho^\vee,\lambda-\mu\rangle},\ \mathcal{A}^\lambda_\mu\simeq\mathbb{A}^{\langle\rho^\vee,\lambda-\mu\rangle},\ \overline{\mathcal{W}}^\lambda_\mu\simeq\mathcal{R}^\lambda_\mu\times\mathcal{A}^\lambda_\mu.$$

In particular we have $\overline{\mathcal{W}}^\lambda_\mu\simeq\mathbb{A}^{2\langle\rho^\vee,\lambda-\mu\rangle}$.

Let $\Delta^\vee_{\mu,-} \subset \Delta^\vee_-$ be a subset of the set of negative roots consisting of α such that $\langle \alpha, \mu \rangle < 0$, we also set $\Delta^\vee_{\mu,+} := -\Delta^\vee_{\mu,-}$. It follows from the proof of Theorem 2.29 that there are natural coordinates $\{y_\beta \,|\, \beta \in \Delta^\vee_{\mu,-}\}$, $\{x_\alpha \,|\, \alpha \in \Delta^\vee_{\mu,+}\}$ on the affine spaces $\mathcal{R}^\lambda_\mu$ and $\mathcal{A}^\lambda_\mu$. Recall also that $\overline{\mathcal{W}}^\lambda_\mu$ is a Coulomb branch of a certain framed quiver gauge theory so it has a symplectic structure defined by some symplectic form ω.

Proposition 2.30
Recall the isomorphism $\overline{\mathcal{W}}^\lambda_\mu \simeq \mathcal{A}^\lambda_\mu \times \mathcal{R}^\lambda_\mu$. The symplectic form ω is $\sum_{\alpha \in \Delta^\vee_{\mu,+}} dx_\alpha \wedge d\hat{y}_{-\alpha}$ for some $\hat{y}_{-\alpha}$ of the form $\hat{y}_{-\alpha} = y_{-\alpha} + P$ with $P \in (y_\beta)^2$, where $(y_\beta)^2$ is the square of the ideal generated by $\{y_\beta,\, \beta \in \Delta^\vee_{\mu,-}\}$.

Proposition 2.31
The character of T acting on $\overline{\mathcal{W}}^\lambda_\mu$ equals to $\sum_{\alpha \in \Delta_-,\, \langle \alpha, \mu \rangle > 0} (q^\alpha + q^{-\alpha})$.

Remark 2.32
Note that the formula for T-character of $\overline{\mathcal{W}}^\lambda_\mu$ does not depend on λ but there is no contradiction here since $\lambda = \mu^+$, the unique dominant element of $W\mu$.

3. Idea of the proof

3.1. **Main steps of the proof.**

Definition 3.33 *(Loop rotation action)*
We have the natural $\mathbb{C}^\times$-action on $\overline{\mathcal{W}}^\lambda_\mu, \mathrm{Gr}_G, Z^\alpha$, which is induced from the following action on $\mathbb{P}^1$: $(x : y) \mapsto (tx : y)$, here $\infty = (0 : 1)$.

It follows from the fact that λ is minuscule that we have

$$\overline{\mathrm{Gr}}^\lambda_G = \mathrm{Gr}^\lambda_G = G \cdot z^\lambda.$$

It follows that $\overline{\mathrm{Gr}}^\lambda_G$ is $\mathbb{C}^\times$-pointwise invariant w.r.t. the loop rotation action.

We have the natural $\mathbb{C}^\times$-equivariant morphism

$$\mathbf{p} \colon \overline{\mathcal{W}}^\lambda_\mu \to \overline{\mathrm{Gr}}^\lambda_G.$$

The following proposition is proved in [Kry18, Theorem 3.1 (1)] (here we do not need λ to be minuscule).

Proposition 3.34
The map $\mathbf{p}$ restricted to $\mathcal{R}^\lambda_\mu$ induces an isomorphism

$$\mathcal{R}^\lambda_\mu \xrightarrow{\sim} U_-(\mathcal{K}) \cdot z^\mu \cap \overline{\mathrm{Gr}}^\lambda_G.$$

Proof. Use modular or matrix descriptions of $\mathcal{R}^\lambda_\mu$. □

We conclude that $\mathcal{R}^\lambda_\mu \subset \overline{\mathcal{W}}^\lambda_\mu$ is $\mathbb{C}^\times$-fixed and is isomorphic to a certain Bruhat cell in $\overline{\mathrm{Gr}}^\lambda_G = G \cdot z^\lambda \simeq G/P_\lambda$, hence, is an affine space.

Lemma 3.35
We have $(\overline{\mathcal{W}}^{\lambda}_{\mu})^{\mathbb{C}^\times} = \mathcal{R}^{\lambda}_{\mu}$.

Proof. Follows from the matrix description of $\overline{\mathcal{W}}^{\lambda}_{\mu}$. ☐

Lemma 3.36
The $\mathbb{C}^\times$*-action on* $\mathcal{A}^{\lambda}_{\mu}$ *via loop rotation contracts it to the point* z^μ *(here we do not need* λ *to be minuscule).*

Proof. Matrix description. ☐

Lemma 3.37 *(Key lemma)*
For a minuscule λ *and* $\mu \in W\lambda$ *the loop rotation action contracts* $\overline{\mathcal{W}}^{\lambda}_{\mu}$ *to* $\mathcal{R}^{\lambda}_{\mu}$.

Proof. Can be deduced from previous lemmas using dimensions estimations and the fact that $\overline{\mathcal{W}}^{\lambda}_{\mu}$ is an affine variety, hence, attractor with respect to the loop rotation is a *closed* subvariety of $\overline{\mathcal{W}}^{\lambda}_{\mu}$. ☐

Corollary 3.38
The image of the morphism $\mathbf{p}\colon \overline{\mathcal{W}}^{\lambda}_{\mu} \to \overline{\mathrm{Gr}}^{\lambda}_{G} = \mathrm{Gr}^{\lambda}_{G}$ *coincides with* $\mathbf{p}(\mathcal{R}^{\lambda}_{\mu}) \simeq \mathcal{R}^{\lambda}_{\mu}$. *In other words the morphism* $\mathbf{p}$ *is just a contraction of* $\overline{\mathcal{W}}^{\lambda}_{\mu}$ *to* $\mathcal{R}^{\lambda}_{\mu} \simeq \mathbb{A}^{\langle \rho^\vee, \lambda-\mu\rangle}$.

It remains to note that there exists a certain involution (to be called Cartan involution, see [BFN16, Section 2(vii)]) $\iota\colon \overline{\mathcal{W}}^{\lambda}_{\mu} \xrightarrow{\sim} \overline{\mathcal{W}}^{\lambda}_{\mu}$ that identifies $\mathcal{R}^{\lambda}_{\mu}$ with $\mathcal{A}^{\lambda}_{\mu}$ so they are both are $\mathbb{A}^{\langle \rho^\vee, \lambda-\mu\rangle}$. Note also that $\mathbf{p}$ is a locally trivial fibration over $\mathcal{R}^{\lambda}_{\mu} = \mathbb{A}^{\langle \rho^\vee, \lambda-\mu\rangle}$ with fiber $\mathcal{A}^{\lambda}_{\mu} = \mathbb{A}^{\langle \rho^\vee, \lambda-\mu\rangle}$, hence, it is trivial and we must have

$$\overline{\mathcal{W}}^{\lambda}_{\mu} \simeq \mathcal{R}^{\lambda}_{\mu} \times \mathcal{A}^{\lambda}_{\mu} \simeq \mathbb{A}^{2\langle \rho^\vee, \lambda-\mu\rangle}.$$

This observation finishes the proof.

Remark 3.39
Using the natural action $U_- \curvearrowright \overline{\mathcal{W}}^{\lambda}_{\mu}$ *the trivialization and isomorphisms above can be made canonical.*

3.2. **Main technical tool – matrix description of (generalized) slices.** For a complex algebraic group H, we set $H[z] := H(\mathbb{C}[z])$, $H[[z^{-1}]] := H(\mathbb{C}[[z^{-1}]])$ and denote by $H[[z^{-1}]]_1$ the kernel (preimage of $1 \in H$) of the natural evaluation at ∞ morphism $\mathrm{ev}_\infty\colon H[[z^{-1}]] \mapsto H$.

In [BFN16, Section 2(xi)], the following isomorphism was constructed:

$$\Psi\colon \overline{\mathcal{W}}^{\lambda}_{\mu} \simeq (U[[z^{-1}]]_1 z^\mu B_-[[z^{-1}]]_1 \bigcap \overline{G[z]z^\lambda G[z]}),$$

where the right hand side is considered as a locally closed subvariety in the ind-scheme $G((z^{-1})) := G(\mathbb{C}((z^{-1})))$.

3.3. Gl_2 **example of slices.** Let $G = \mathrm{GL}_2$, $\lambda = (N, 0) = N\omega_1$ and $\mu = (N-m, m) = N\omega_1 - m\alpha_1$. It follows from [BFN16, Section 2(xii)] that $\overline{\mathcal{W}}^{\lambda}_{\mu}$ identifies with the space

$\mathcal{M}_\mu^\lambda \subset \mathrm{Mat}_{2\times 2}[z]$ of matrices $\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that A is a monic polynomial of degree m, while the degrees of B and C are strictly less than m, and $\det \mathbf{M} = z^N$.

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