joint with Lev Rozansky

Thm $HHH_{alg}(\beta) = HHH_{geo}(\beta)$ $\quad \beta \in Br_n$

3gr. vect. spaces

HHH_{alg}: Φ_R (Rouquier '04) $: Br_n \to H_0(SBim_n)$

homotopy cat.

Hochschild. homology

Khovanov - Rozansky '05 $\quad HHH_{alg}(\beta) = H_\bullet(\underset{t}{}\, HH_{\underset{a}{*}}(\Phi_R(\beta)))$

HHH_{geo}: $\quad Br_n \xrightarrow[\text{'16}]{\Phi} MF^{st}_n \xrightarrow[\text{'19}]{CH} \boxed{D^{per}_{T_{qt}}(Hilb_n(\mathbb{C}^2))}$ $\quad$ (q)

$Br_n \xrightarrow{Jr} D^{per}_{T_{qt}}(Hilb_n(\mathbb{C}^2))$

O - Rozansky '16, '19

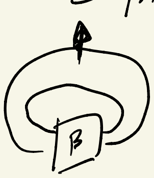
$T_{q,t} = \mathbb{C}^*_{q^2} \times \mathbb{C}^*_{t^2/q^2}$

$HHH_{geo}(\beta) = \mathbb{H}(\mathcal{O} \otimes \Lambda^{\overset{a}{\bullet}} \mathcal{B}, Jr(\beta))$

taut.

Cor $L(\beta)$ is knot $\quad \Rightarrow \quad HHH_{alg}(\beta) = HHH_{alg}(\beta)\Big|_{q \to t/q}$

$\pi_0(L(\beta)) = 1$



Also see recent result by Gorsky - Hogancamp Mellit.

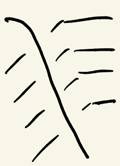
$\underline{Jr(\beta \cdot FT) = Jr(\beta) \otimes \det(\mathcal{B})}$; $\quad Hilb_n(\mathbb{C}^2) = \{ I \subset \mathbb{C}[x,y] \text{ ideal}$

full twist $\quad \mathbb{C}[x,y]/I = \mathbb{C}^n \}$

$\check{\mathcal{B}}\Big|_I = \mathbb{C}[x,y]/I$

Cor $\quad \beta = cox = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \in Br$ $\quad Jr(\underline{cox \cdot FT^d}) = Jr(cox) \otimes \det(\mathcal{B})^d$

$Jr(cox) = \mathcal{O}_{Hilb(\mathbb{C}^2, 0)}$

cox 

Hogancamp -Mellit

$Hilb_n(\mathbb{C}^2, 0) \subset Hilb_n(\mathbb{C}^2)$

$supp(\mathbb{C}[x,y]/I) = (0,0)^n$.

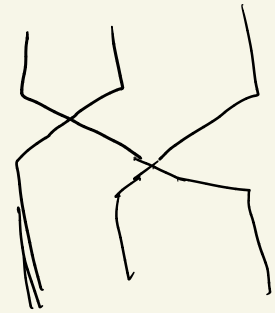
$HHH_{alg}(T_{n,1+nd}) = H^*(\mathcal{O}_{Hilb_n(\mathbb{C}^2,0)} \otimes \det(\mathcal{B})^d)$.

Thm [OR] $MF_n^b \subset MF_n^{st}$ additive.

$B: MF_n^b \longrightarrow SBim_n$.

B is fully faithful equiv.

Br_n^b are graph braids:



$\gamma \in Br_3^b$

Br_n^b is monoid with relations similar to Hecke alg rels.

Soergel '01.

$SBim_n$: $\quad B_i = R_n \underset{R_n^{s_{i\,i+1}}}{\otimes} R_n \qquad s_{i\,i+1}(x_i) = x_{i+1}$, $\; s_{i\,i+1}(x_{i+1}) = x_i$

$\cap$

Bim_n bimodules monoidal. over $R_n = \mathbb{C}[x_1,\dots,x_n]$ $\quad \deg x_i = q$

$SBim_n$ is spanned by direct summands of the products of $B_1,\dots,B_{n-1}$

$n!$ simple objects $\quad K(SBim_n) = H_q(S_n)$.

Thm

$$\begin{tikzcd} & Br_n^b \arrow[dl, "\Phi^b"'] \arrow[dr, "\Phi_S"] & \\ MF_n^b \cap MF_n^{st} \arrow[rr, "B"] \arrow[drr] & & SBim_n \\ & & Ho(Bim_n) \end{tikzcd}$$

Soergel '01 MOY $\Rightarrow$

$$\begin{tikzcd} & Br_n \arrow[dl, "\Phi"'] \arrow[dr, "\Phi_R"] & \\ Ho(MF_n^b) \arrow[rr, "B"] & & Ho(SBim_n) \\ MF_n \arrow[urr, "B"'] & & \end{tikzcd}$$

Rouquier

Murakami-Ohtsuki-Yamada '98

Conj $\boxed{Ho(MF_n^b) = MF_n^{st}}$.

$\Phi^b : Br_n^b \to MF_n^b$. $\Phi^b(\gamma)$ are pure.

purity $\quad$ $Hom(\Phi^b(\gamma), \Phi^b(1))$ is supported on one t-degree.

$$\boxed{(\Phi^b(\gamma))^\vee = \Phi^b(\gamma)}$$

$MF(Z,F)$: Obj : $M_0 \xrightarrow{D_0} M_1 \xrightarrow{D_1} M_0 \xrightarrow{D_0} M_1$

Z is affine. quasi-affine. $\quad$ M_i are free over $\mathbb{C}[Z]$.

$D^2 = F$

Orlov '01 $\quad$ $MF(Z,F)$ is a triangulated.

$$MF\left(Z \times \mathbb{C}^n_y, \sum_{i=1}^n f_i(z)\, y_i\right) = DG(Z_f \subset Z)$$

Koszul $\quad$ per. $\quad$ $Z_f = \{f_1 = \dots = f_n = 0\}$.

$$MF_n = MF_{GL_n}(\mathfrak{g} \times \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}, W)$$

$X \quad Z_1 \quad Z_2$ $\qquad$ $\mathfrak{g} = \mathfrak{gl}_n$. $\quad G = GL_n$.

$\tilde{\mathfrak{g}} = G \times \mathfrak{b} / B \xrightarrow{\mu} Ad_g(Y) \in \mathfrak{g}$.

g, Y

$$W = Tr(X(\mu(z_1) - \mu(z_2)))$$

$$\boxed{MF_n = MF_{G\times \boxed{B^2}}(\underset{X}{\mathfrak{g}} \times \underset{g_1}{G} \times \underset{Y_1}{\mathfrak{b}} \times \underset{g_2}{G} \times \underset{Y_2}{\mathfrak{b}},\ W)}$$

$$W(\ldots) = \mathrm{Tr}(X(\mathrm{Ad}_{g_1} Y_1 - \mathrm{Ad}_{g_2} Y_2))$$

linear on X

$$MF_n = DG(\mathring{S}t) \xleftarrow[\text{Riche}]{\text{Bezrukavnikov}} Br_n^{aff}.$$

$$\uparrow \Phi \qquad Br_n^{aff} \qquad\qquad \begin{matrix} \Uparrow \\ \underset{z_2}{\tilde{\mathfrak{g}}} \times \underset{z_1}{\tilde{\mathfrak{g}}} \end{matrix} \qquad \mu(z_1) = \mu(z_2)$$

$$MF_n^{st} = MF_{G\times B^2}\left(\left((\mathcal{X}) \times \mathbb{C}^n\right)^{st}, W\right) \qquad \mathcal{X} = \mathfrak{g} \times (G \times \mathfrak{b})^2$$

$\uparrow$ OR.

Br_n

X, g_1, Y_1, g_2, Y_2, v st: $\boxed{\begin{matrix} \mathbb{C}\langle \boxed{X}, \mathrm{Ad}_{g_1} Y_1 \rangle v = \mathbb{C}^n \\ \mathbb{C}\langle X, \mathrm{Ad}_{g_2} Y_2 \rangle v = \mathbb{C}^n \end{matrix}}$

$$\mathfrak{g} \times (G \times \mathfrak{b})^3 \xrightarrow{\pi_{ij}} \mathfrak{g} \times (G \times \mathfrak{b})^2.$$

$$\mathcal{F} * \mathcal{G} = \pi_{13*}(\pi_{12}^*(\mathcal{F}) \otimes \pi_{23}^*(\mathcal{G})).$$

$$\mathbb{B} : MF_n^{st} \to Ho(Bim)$$

$$\begin{tikzcd}
 & (\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \times \mathbb{C}^n)^{st} \arrow[dl, leftrightarrow, "\cdot z"'] \arrow[d, "res"'] \arrow[dr, "\lambda \times \lambda"] & \\
(0,z). & (\mathfrak{g} \times \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \times \mathbb{C}^n)^{st} & \mathfrak{h} \times \mathfrak{h}.
\end{tikzcd}$$

$$\underset{X}{\mathfrak{g}} \times \underset{(g_1, Y_1)}{\tilde{\mathfrak{g}}} \times \underset{(g_2, Y_2)}{\tilde{\mathfrak{g}}} \times \mathbb{C}^n \;\text{"}\; \mathcal{X}^{st}/B^2$$

$$W = Tr(X(\dots))$$

$\lambda : \mathfrak{b} \to \mathfrak{h}$ extract the diag.

$$\mathbb{B} = (\lambda \times \lambda)_* \circ res^*$$

Simplest model.

$$\mathcal{X} = \mathfrak{g} \times G \times \mathfrak{b} \times G \times \mathfrak{b}.$$

$G \circlearrowright$ freely.

$$\mathcal{X}^0 = \underset{X}{\mathfrak{g}} \times \overset{Y_1}{\mathfrak{b}} \times \overset{g_{12}}{G} \times \overset{Y_2}{\mathfrak{b}} = \mathcal{X}/G.$$

$$W^0(--) = Tr(X(Y_1 - Ad_{g_{12}} Y_2)) =$$

$$= \boxed{Tr(\underbrace{X_+}_{\text{upper } \Delta \text{ part.}}(Y_1 - Ad_{g_{12}} Y_2))} + \underline{\text{quadratic}}$$

W^0

$$MF_n \simeq \overline{MF}_n^0 = MF_{B^2}(\underset{X_+}{\mathfrak{b}} \times \underset{\substack{? \\ \lambda(Y_1)}}{\mathfrak{h}} \times \underset{g_{12}}{G} \times \underset{Y_2}{\mathfrak{b}}, W)$$

$$\left(\underset{X}{\mathfrak{b}} \times \mathfrak{h} \times \underset{g_{12}}{G} \times \underset{Y_1}{\mathfrak{b}} \times \underset{v}{\mathbb{C}^n} \right)^{st} \supset B^2.$$

$$\uparrow i_x.$$

$$0 \times \mathbb{C}^n \times \boxed{U_-} \times \mathbb{C}^n \times \{e_n\}$$

U_- lower triangular group.

$$i_x(y_1, g, y_2) = (0, \mathfrak{h}(y_1), g, \underbrace{\mathfrak{h}(y_2) + J}_{\text{regular}}, \cancel{e_n})$$

$$\boxed{B = \pi_* \circ i_x^*}$$

$B(MF_n^b) \Rightarrow SBim_n$

$$\boxed{MF\left(\underset{X}{\mathfrak{g}} \times \underset{g}{G} \times \underset{Z}{\mathfrak{g}}, \mathrm{Tr}(X[Z - \mathrm{Ad}_g Z]) \right)}$$

~~BELOW IS THE DISCUSSION~~ ~~END OF LECTURE~~

$$\lambda^{-1} \times \lambda^{-1}(\Gamma_w)$$

$$\widetilde{St} = \bigsqcup_{w \in S_n} St_w$$

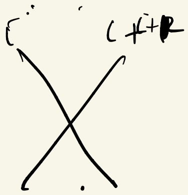
$\mathfrak{gl}_n \supset GL_n$

$\xrightarrow{\lambda \times \lambda} \mathfrak{h} \times \mathfrak{h}$

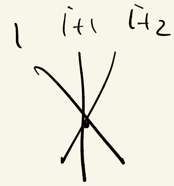
$$g_1\, Y_1 \qquad g_2\, Y_2$$

$$\mathrm{Ad}_{g_1} Y_1 = \mathrm{Ad}_{g_2} Y_2.$$

DG :



$\mathcal{O}_{\widetilde{St}_1 \cup \widetilde{St}_{s_{i\,i+1}}}$



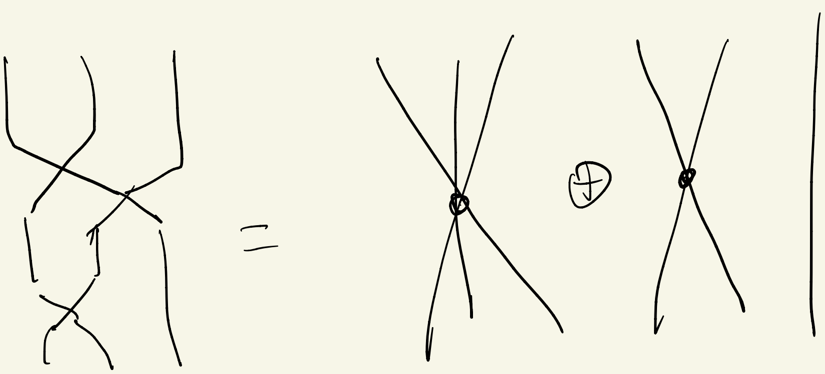
$n=2$

$\mathcal{O}_{St_w}$

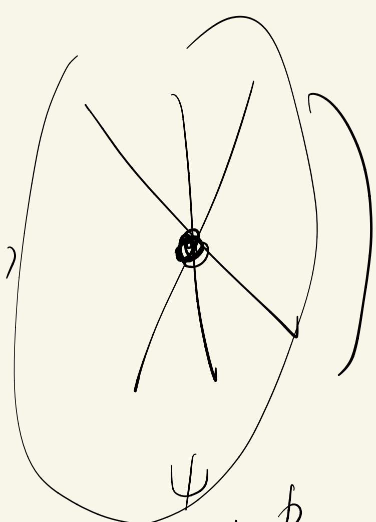
$w \in S_3(i, i+1, i+1)$

$$\mathcal{O}_{\widetilde{St}_1 \cup \widetilde{St}_{s_1}} = \mathcal{O}_{\widetilde{St}}$$

$$DG: \mathcal{O}_{\widetilde{St}} * \mathcal{O}_{\widetilde{St}} = \mathcal{O}_{\widetilde{St}} \oplus q^2 \mathcal{O}_{\widetilde{St}}$$



$B :=$ $\mathrm{Hom}\big(-,$



$\big)$

$\mathcal{O}(1)$

$MF_n^b = \mathcal{O}_{\widetilde{St}}$

$MF_2(\ldots) = Coh(\mathbb{P}^1)$ ∞

$MF_2^{St}(\ldots) = Coh(\mathbb{P}^1)$ ∞

$Br_1^{aff} = \mathbb{Z}$ $\qquad$ $Br^{lin} = 1.$

$MF_{\mathbb{C}^*}(\mathbb{C}, 0) \xrightarrow[\text{vs}]{\sim} MF_{\mathbb{C}^*}(\mathbb{C}^*, 0) \qquad n = 1$

$x \quad 1$