UC Davis–Eugene

Algebra & Discrete Mathematics Seminar

Incompressible symmetric tensor categories

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(jt with Dave Benson and Pavel Etingof)

Symmetric tensor categories

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Representation categories

Base field: $k = \bar{k}$

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Thus $\mathrm{Rep}(G)$ is a rigid symmetric tensor category equipped with a (symmetric) tensor functor to Vec

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Theorem (Grothendieck, Saavedra Rivano, Deligne-Milne)

Assume $\mathcal{C}$ is Tannakian. Then $\mathcal{C} = Rep(G)$ for some (unique) affine group scheme G.

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super Tannakian theory and generalization

Theorem (Deligne)

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Question: What are categories which can't be expressed in terms of "group theory" in smaller categories?
Equivalently, which categories do not admit $\otimes$ functors to smaller categories?

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Any more examples?

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Conjecture: No more incompressible categories in characteristic zero.

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Theorem (U. Jannsen)

Assume $\dim Hom(X, Y) < \infty$ *and any nilpotent endomorphism in* $\mathcal{T}$ *has trace zero. Then* $\overline{\mathcal{T}}$ *is semisimple (and so abelian). Moreover*

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Remark: assume $F : \mathcal{T} \to \mathcal{C}$ is a $\otimes$ functor to abelian $\mathcal{C}$. Then any nilpotent endomorphism in $\mathcal{T}$ has trace zero.

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This implies: $L_{p-1} \otimes L_{p-1} = \mathbf{1}$ and $\langle \mathbf{1}, L_{p-1} \rangle = \mathrm{sVec}$.

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Simple objects $L_1 = \mathbf{1}, L_2, \ldots, L_{p-1}$
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This implies: $L_{p-1} \otimes L_{p-1} = \mathbf{1}$ and $\langle \mathbf{1}, L_{p-1} \rangle = \text{sVec}$.
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Technology: Hopf algebras (in categories) and graded extensions

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Exercise. Let P be a projective object and $f : X \to Y$ be any morphism Then $\mathrm{id}_P \otimes f : P \otimes X \to P \otimes Y$ is split

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Wanted: more examples of splitting tensor ideals!

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$[23045]_p \rightsquigarrow 2(-3)0(-4)5 = 2p^4 - 3p^3 - 4p + 5$

Descendants of i: all positive numbers you get in this way

E.g. $[23045]_p$ has $2^3 = 8$ descendants

Tubbenhauer-Wedrich: $C_{ij} = |\{\text{descendants of } i\} \cap \{\text{descendants of } j\}|$

Exercise: $C_{ij} = 0$ or power of 2

Exercise: $\det C_{ij}$ = power of p

Embeddings

We have $\mathrm{Ver}_p \subset \mathrm{Ver}_{p^2} \subset \mathrm{Ver}_{p^3} \subset \dots$

For $p > 2$, $\mathrm{Ver}_{p^n} = \mathrm{Ver}_{p^n}^+ \boxtimes \mathrm{sVec}$

More properties of Ver_{p^n}

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Corollary: $C = DD^T$ where D is the decomposition matrix

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Universality

Let $\mathrm{Ver}_{p^\infty} = \cup_n \mathrm{Ver}_{p^n}$

Let $\mathcal{C}$ be a pre-Tannakian category of sub-exponential growth

Question 4. Is there an exact tensor functor $\mathcal{C} \to \mathrm{Ver}_{p^\infty}$?

Thanks for listening!