



# New approaches to the restriction problem

Digjoy Paul, The Institute of Mathematical Sciences (HBNI), Chennai, India.

Joint work with Sridhar Narayanan, Amritanshu Prasad and Shraddha Srivastava.

Algebra and Discrete Mathematics, UC DAVIS, 26th May, 2020.

#### This talk is based on

- S. Narayanan, D. Paul, A. Prasad, S. Srivastava Polynomial Induction and the Restriction Problem, (submitted) 2020, arxiv:2004.03928.
- S. Narayanan, D. Paul, A. Prasad, S. Srivastava

  Character Polynomials and the Restriction Problem,
  (submitted) 2020, arXiv:2001.04112.

#### Plans for the talk

- 1. Restriction problem: an open problem.
- 2. History: attempts so far.
- 3. Results obtained using new approaches.
- 4. First approach: character polynomials.
- 5. Second approach: polynomial induction.

# Polynomial representation

#### Polynomial representation

A pair  $(\rho, W)$  where  $\rho: GL_n(\mathbf{C}) \to GL(W)$  is a group homomorphism such that the entries of  $\rho(A)$  are **polynomials** in the entries of  $A \in GL_n(\mathbf{C})$ .

#### **Example**

$$\rho: \mathit{GL}_2(\mathbf{C}) o \mathit{GL}_3(\mathbf{C})$$
 given by

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}.$$

# Weyl modules and Schur polynomials

### Irreducible representations

**Weyl modules:**  $\{W_{\lambda}(n) : len(\lambda) \le n\}$  has dimension =

 $|SSYT(\lambda, \leq n)|$ 

Characters: Schur polynomials

$$char(W_{\lambda}(n)) = trace(\rho(diag(x_1, \dots, x_n)); W_{\lambda}(n)) = s_{\lambda}(x_1, \dots, x_n)$$

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#### Goal

To understand the decomposition of the restriction of a polynomial representation of  $GL_n(\mathbf{C})$  to the subgroup  $S_n$ :

$$\mathrm{Res}_{S_n}^{GL_n(\mathbf{C})}W_{\lambda}(n)\cong \bigoplus_{\mu\vdash n}V_{
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**Open problem:** Positive combinatorial interpretation for the multiplicities  $r_{\lambda,\mu}$ .

#### Littlewood's formula

$$r_{\lambda,\mu} = \langle s_{\lambda}, s_{\mu}[1 + h_1 + h_2 + \cdots] \rangle.$$

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#### Mike Zabrocki, OPAC 2021

"This is an advance in the problem, but recasts the solution of one problem in terms of another for which we don't have a combinatorial formula".

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Nate Harman "Representations of monomial matrices and restriction from  $GL_n$  to  $S_n$ ".

# Theorem (Orellana –Zabrocki)

Let 
$$\lambda = (\lambda_1, \dots, \lambda_s)$$
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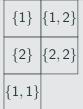
 $SSMT(\mu, \lambda)$  denote the set of semistandard multiset tableaux of shape  $\mu$  and content  $\{1^{\lambda_1}, \dots, 1^{\lambda_s}\}$ .

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# Example

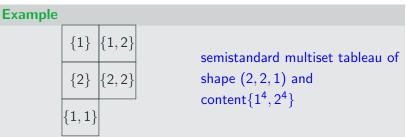


semistandard multiset tableau of shape (2,2,1) and  $\mathsf{content}\{1^4,2^4\}$ 

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 $SSMT(\mu,\lambda)$  denote the set of semistandard multiset tableaux of shape  $\mu$  and content  $\{1^{\lambda_1},\dots,1^{\lambda_s}\}$ .



**Question:** What is the generating function for  $SSMT(\mu, \lambda)$ ?

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- 3. If  $\lambda = (a+1,1^b)$  then  $r_{\lambda,(n)} > 0$  if and only if  $a \ge {b+1 \choose 2}$ .

Character polynomial unifies characters of  $S_n$  across all n.

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# **Example (Standard representations)**

 $trace(w; V_{(n-1,1)}) = no.$  of fixed points of  $w - 1 = X_1(w) - 1$ .

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Character polynomial is a polynomial in the cycle-counting functions.

Note that P is a graded algebra when the variable  $X_i$  has degree i.

# Character polynomials ctd.

# Theorem (Binomial basis)

Given a partition 
$$\alpha=1^{a_1}2^{a_2}\cdots$$
, define  $\binom{X}{\alpha}:=\prod_{i\geq 1}\binom{X_i}{a_i}$ . Then  $\left\{\binom{X}{\alpha}\mid \alpha \text{ is a partition }\right\}$  is a basis of  $P$ .

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#### Representations with Polynomial Character

A family of representation  $\{V_n\}$  of  $S_n$  is said to have *eventually* polynomial character if there exists  $q \in P$  and a positive integer N such that, for each  $n \geq N$  and each  $w \in S_n$ ,

$$trace(w; V_n) = q(X_1(w), X_2(w), ...)$$

**Church, Ellenberg and Farb**, "Fl-modules and stability for representations of symmetric groups".

# Character polynomial of Weyl modules

Specht module has eventually polynomial character, that is, for every partition  $\lambda$  there exist  $q_{\lambda} \in P$  such that

$$trace(w; V_{n-|\lambda|,\lambda}) = q_{\lambda}(X_1(w), X_2(w), \dots).$$

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#### Goal

To compute  $\mathcal{S}_{\lambda} \in P$  such that

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#### Goal

To compute  $S_{\lambda} \in P$  such that

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Recipe is to find character polynomial of  $\mathrm{Sym}^d(\mathbf{C}^n)$  or  $Alt^d(\mathbf{C}^n)$  and apply Jacobi–Trudi identities.

# **Character Polynomials of** *Sym* **and** *Alt*

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#### Theorem (NPPS)

$$H_d = \sum_{\alpha \vdash d} \begin{pmatrix} X \\ \alpha \end{pmatrix},$$

$$E_d = \sum_{\alpha \vdash d} (-1)^{a_2 + a_4 + \dots} \begin{pmatrix} X \\ \alpha \end{pmatrix}.$$

Here 
$$\binom{n}{d} = \binom{n+d-1}{d}$$
 and  $\binom{X}{\alpha} := \prod_{i \geq 1} \binom{X_i}{a_i}$  when  $\alpha = 1^{a_1} 2^{a_2} \cdots$ .

Recall  $S_{\lambda} = det(H_{\lambda_i+j-i}) = det(E_{\lambda'_i+j-i}).$ 

Both  $q_{\lambda}$  and  $\mathcal{S}_{\lambda}$  are inhomogeneous polynomials of degree  $|\lambda|$  in the graded algebra P.

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#### Two bases of P

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#### Observation

$$S_{\lambda} = \sum r_{\lambda,\mu[n]} q_{\mu}$$

where  $\mu[n] := (n - |\mu|, \mu_1, \mu_2, \ldots)$ .

# Relation to symmetric functions

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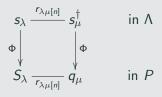
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### Character polynomials and symmetric functions



# Moment of a character polynomial

For two representations V and W of  $S_n$ , we have

$$\langle V, W \rangle_n := dim Hom_{S_n}(V, W) = \frac{1}{n!} \sum_{w \in S_n} trace(w; V) trace(w; W)$$

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Can be extended to this setting:

#### **Definition (Moment)**

Given character polynomials  $q_1, q_2$  corresponding to two families of  $S_n$  representations  $\{V_n\}, \{W_n\}$  respectively, define moment

$$\langle q_1 q_2 \rangle_n := \sum_{\alpha \vdash n} \frac{1}{z_{\alpha}} q_1(\alpha) q_2(\alpha).$$

$$\alpha = 1^{a_1} 2^{a_2} \cdots \qquad z_{\alpha} = \prod_i i^{a_i} a_i!.$$

#### Moments and the restriction coefficients

#### Theorem (NPPS)

For every integer partition  $\alpha = 1^{a_1}2^{a_2}\cdots$ , we have:

$$\left\langle \begin{pmatrix} X \\ \alpha \end{pmatrix} \right\rangle_n = \begin{cases} 0 & \text{if } n < |\alpha|, \\ 1/z_\alpha & \text{otherwise.} \end{cases}$$

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Moments of a character polynomial stablises beyond a certain n. Recall

- $ResW_{\lambda} \cong \bigoplus_{\mu} V_{\mu[n]}^{r_{\lambda\mu[n]}}$   $S_{\lambda} = \sum_{\lambda} r_{\lambda,\mu[n]} q_{\mu}$ .

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- $S_{\lambda} = \sum r_{\lambda,\mu[n]} q_{\mu}$ .
- By definition,  $r_{\lambda,\mu[n]} = \langle \mathcal{S}_{\lambda} q_{\mu} \rangle$ .
- After expanding the product in the binomial basis, the moment can be computed and hence restriction coefficients.

# Moment of Weyl character polynomial

### Theorem (NPPS)

For every partition  $\lambda$ ,  $\langle S_{\lambda} \rangle_n$  is the coefficient of  $t^{\lambda}v^n$  in

$$\prod_{i < j} (1 - t_j/t_i) \prod_{R \subset [I]} (1 - t^R v)^{-1}.$$

Let  $p_{\leq n}(X)$ ) denote the number of vector partitions of X with at most n parts.

### Theorem (NPPS)

For every partition  $\lambda = (\lambda_1, \cdots, \lambda_l)$ ,

$$r_{\lambda,(n)} = \sum_{w \in S_I} sgn(w) p_{\leq n}(\lambda_1 - 1 + w(1), \cdots, \lambda_I - I + w(I)).$$

# Why polynomial induction?

- $r_{\lambda,\mu} = \dim \operatorname{\mathsf{Hom}}_{S_n}(\operatorname{Res}W_\lambda, V_\mu) = \langle s_\lambda, s_\mu[1+h_1+h_2+\cdots] \rangle$
- Scharf-Thibon gave a proof using Hopf algebra techniques

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- Scharf-Thibon gave a proof using Hopf algebra techniques
- The Frobenius reciprocity suggests

#### Question

Does there exist an induction functor  $\operatorname{Ind}^d:\operatorname{Rep} S_n \to \operatorname{Rep}^d GL_n$  such that

$$\operatorname{\mathsf{Hom}}_{\mathcal{S}_n}(\operatorname{Res} W_\lambda,\,V_\mu)\cong\operatorname{\mathsf{Hom}}_{\mathit{GL}_n}(W_\lambda,\operatorname{\mathsf{Ind}}^dV_\mu)$$

## **Definition (Frobenius)**

Given a representation  $(\pi, V)$  of H,  $H \leq G$ , can define  $(\pi^G, \operatorname{Ind}_H^G V) \in \operatorname{Rep} G$ 

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#### Our work

We adapt Mackey's construction to the setting of polynomial representations.

**Notation:**  $M_n$ : Ring of  $n \times n$  matrices with entries in **C**.  $P^d(M_n)$ : Space of homogeneous polynomials of degree d in the entries of matrices  $Q \in M_n$ .

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#### **Example**

Let  $1_n$  denote the trivial representation of  $S_n$ .

$$\operatorname{Ind}^d 1_n = \{ f \in P^d(M_n) \mid f(wQ) = f(Q) \text{ for all } w \in S_n, \ Q \in M_n \}^{21/23}$$

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