New approaches to the restriction problem

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Joint work with Sridhar Narayanan, Amritanshu Prasad and Shraddha Srivastava.

Algebra and Discrete Mathematics, UC DAVIS, 26th May, 2020.

This talk is based on

S. Narayanan, D. Paul, A. Prasad, S. Srivastava
Polynomial Induction and the Restriction Problem,
(submitted) 2020, arxiv:2004.03928.

S. Narayanan, D. Paul, A. Prasad, S. Srivastava
Character Polynomials and the Restriction Problem,
(submitted) 2020, arXiv:2001.04112.

Plans for the talk

1. Restriction problem: an open problem.
2. History: attempts so far.
3. Results obtained using new approaches.
4. First approach: character polynomials.
5. Second approach: polynomial induction.

Polynomial representation

A pair (ρ, W) where $\rho : GL_n(\mathbf{C}) \to GL(W)$ is a group homomorphism such that the entries of $\rho(A)$ are **polynomials** in the entries of $A \in GL_n(\mathbf{C})$.

Example

$\rho : GL_2(\mathbf{C}) \to GL_3(\mathbf{C})$ given by

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}.$$

Weyl modules and Schur polynomials

Irreducible representations

Weyl modules: $\{W_\lambda(n) : len(\lambda) \le n\}$ has dimension $=$ $|SSYT(\lambda, \le n)|$

Characters: Schur polynomials

$$char(W_\lambda(n)) = trace(\rho(\mathrm{diag}(x_1, \dots, x_n)); W_\lambda(n)) = s_\lambda(x_1, \dots, x_n)$$

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$x_1^2 x_2$	$x_1^2 x_3$	$x_1 x_2^2$	$x_1 x_2 x_3$	$x_1 x_2 x_3$	$x_1 x_3^2$	$x_2^2 x_3$	$x_2 x_3^2$

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To understand the decomposition of the restriction of a polynomial representation of $GL_n(\mathbf{C})$ to the subgroup S_n:

$$\mathrm{Res}_{S_n}^{GL_n(\mathbf{C})} W_\lambda(n) \cong \bigoplus_{\mu \vdash n} V_\nu^{\oplus r_{\lambda,\mu}}.$$

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Open problem: Positive combinatorial interpretation for the multiplicities $r_{\lambda,\mu}$.

Attempts so far

Littlewood's formula

$$r_{\lambda,\mu} = \langle s_\lambda, s_\mu[1 + h_1 + h_2 + \cdots]\rangle.$$

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Mike Zabrocki, OPAC 2021

"This is an advance in the problem, but recasts the solution of one problem in terms of another for which we don't have a combinatorial formula".

Attempts so far ctd.

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Nate Harman "Representations of monomial matrices and restriction from GL_n to S_n".

Restriction in a few special cases

Theorem (Orellana –Zabrocki)

Let $\lambda = (\lambda_1, \ldots, \lambda_s)$*, consider the* S_n*−module*

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$SSMT(\mu, \lambda)$ denote the set of semistandard multiset tableaux of shape μ and content $\{1^{\lambda_1}, \ldots, 1^{\lambda_s}\}$.

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Question: What is the generating function for $SSMT(\mu, \lambda)$?

Positivity of some Restriction coefficients

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For which λ is $r_{\lambda,(n)} = \langle \mathrm{Res} W_\lambda(n), V_{(n)} \rangle_{S_n} > 0$?

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2. *If λ has two columns then $r_{\lambda,(n)} > 0$ if and only if $\lambda_1' - \lambda_2' \leq 1$.*
3. *If $\lambda = (a+1, 1^b)$ then $r_{\lambda,(n)} > 0$ if and only if $a \geq \binom{b+1}{2}$.*

Character polynomial

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Character polynomial is a polynomial in the cycle-counting functions.

Note that P is a graded algebra when the variable X_i has degree i.

Character polynomials ctd.

Theorem (Binomial basis)

Given a partition $\alpha = 1^{a_1} 2^{a_2} \cdots$ *, define* $\binom{X}{\alpha} := \prod_{i \geq 1} \binom{X_i}{a_i}$. *Then* $\left\{ \binom{X}{\alpha} \mid \alpha \text{ is a partition } \right\}$ *is a basis of* P.

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Representations with Polynomial Character

A family of representation $\{V_n\}$ of S_n is said to have *eventually polynomial character* if there exists $q \in P$ and a positive integer N such that, for each $n \geq N$ and each $w \in S_n$,

$$trace(w; V_n) = q(X_1(w), X_2(w), \dots)$$

Church, Ellenberg and Farb, “FI-modules and stability for representations of symmetric groups”.

Character polynomial of Weyl modules

Specht module has eventually polynomial character, that is, for every partition λ there exist $q_\lambda \in P$ such that

$$\mathit{trace}(w; V_{n-|\lambda|,\lambda}) = q_\lambda(X_1(w), X_2(w), \dots).$$

for $n \geq \lambda_1 + |\lambda|$.

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To compute $\mathcal{S}_\lambda \in P$ such that

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Recipe is to find character polynomial of $\mathrm{Sym}^d(\mathbf{C}^n)$ or $Alt^d(\mathbf{C}^n)$ and apply Jacobi–Trudi identities.

Character Polynomials of *Sym* and *Alt*

Let us find $H_d(w) := trace(w; \mathrm{Sym}^d(\mathbf{C}^n))$ and $E_d(w) := trace(w; Alt^d(\mathbf{C}^n))$.

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Theorem (NPPS)

$$H_d = \sum_{\alpha \vdash d} \left(\!\!\binom{X}{\alpha}\!\!\right),$$

$$E_d = \sum_{\alpha \vdash d} (-1)^{a_2 + a_4 + \cdots} \binom{X}{\alpha}.$$

Here $\left(\!\binom{n}{d}\!\right) = \binom{n+d-1}{d}$ and $\left(\!\!\binom{X}{\alpha}\!\!\right) := \prod_{i \geq 1} \left(\!\!\binom{X_i}{a_i}\!\!\right)$ when $\alpha = 1^{a_1} 2^{a_2} \cdots$.

Stable restriction coefficients

Recall $\mathcal{S}_\lambda = det(H_{\lambda_i+j-i}) = det(E_{\lambda'_i+j-i})$.
Both q_λ and $\mathcal{S}_\lambda$ are inhomogeneous polynomials of degree $|\lambda|$ in the graded algebra P.

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Observation

$$\mathcal{S}_\lambda = \sum r_{\lambda,\mu[n]} q_\mu$$

where $\mu[n] := (n - |\mu|, \mu_1, \mu_2, \ldots)$.

Relation to symmetric functions

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Character polynomials and symmetric functions

$$\begin{array}{ccccc} s_\lambda & \overset{r_{\lambda\mu[n]}}{\rule{2em}{0.4pt}} & s^\dagger_\mu & \qquad & \text{in } \Lambda \\ \Phi \downarrow & & \downarrow \Phi & & \\ S_\lambda & \underset{r_{\lambda\mu[n]}}{\rule{2em}{0.4pt}} & q_\mu & \qquad & \text{in } P \end{array}$$

Moment of a character polynomial

For two representations V and W of S_n, we have

$$\langle V, W \rangle_n := dimHom_{S_n}(V, W) = \frac{1}{n!} \sum_{w \in S_n} trace(w; V) trace(w; W)$$

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Can be extended to this setting:

Definition (Moment)

Given character polynomials q_1, q_2 corresponding to two families of S_n representations $\{V_n\}, \{W_n\}$ respectively, define moment

$$\langle q_1 q_2\rangle_n := \sum_{\alpha\vdash n} \frac{1}{z_\alpha} q_1(\alpha) q_2(\alpha).$$

$$\alpha = 1^{a_1} 2^{a_2} \cdots \qquad z_\alpha = \prod_i i^{a_i} a_i!.$$

Moments and the restriction coefficients

Theorem (NPPS)

For every integer partition $\alpha = 1^{a_1} 2^{a_2} \cdots$, *we have:*

$$\left\langle \binom{X}{\alpha} \right\rangle_n = \begin{cases} 0 & \text{if } n < |\alpha|, \\ 1/z_\alpha & \text{otherwise.} \end{cases}$$

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Moments of a character polynomial stablises beyond a certain n.

Recall

- $ResW_\lambda \cong \bigoplus_\mu V_{\mu[n]}^{r_{\lambda\mu[n]}}$
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- $ResW_\lambda \cong \bigoplus_\mu V_{\mu[n]}^{r_{\lambda\mu[n]}}$
- $\mathcal{S}_\lambda = \sum r_{\lambda,\mu[n]} q_\mu$.
- By definition, $r_{\lambda,\mu[n]} = \langle \mathcal{S}_\lambda q_\mu \rangle$.
- After expanding the product in the binomial basis, the moment can be computed and hence restriction coefficients.

Moment of Weyl character polynomial

Theorem (NPPS)

For every partition λ, $\langle \mathcal{S}_\lambda \rangle_n$ is the coefficient of $t^\lambda v^n$ in

$$\prod_{i<j}(1 - t_j/t_i) \prod_{R \sqsubset [l]} (1 - t^R v)^{-1}.$$

Let $p_{\leq n}(X))$ denote the number of vector partitions of X with at most n parts.

Theorem (NPPS)

For every partition $\lambda = (\lambda_1, \cdots, \lambda_l)$,

$$r_{\lambda,(n)} = \sum_{w \in S_l} sgn(w) p_{\leq n}(\lambda_1 - 1 + w(1), \cdots, \lambda_l - l + w(l)).$$

Why polynomial induction?

- $r_{\lambda,\mu} = dim\,\mathrm{Hom}_{S_n}(\mathrm{Res} W_\lambda, V_\mu) = \langle s_\lambda, s_\mu[1 + h_1 + h_2 + \cdots] \rangle$
- **Scharf–Thibon** gave a proof using Hopf algebra techniques

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- **Scharf–Thibon** gave a proof using Hopf algebra techniques
- The Frobenius reciprocity suggests

Question

Does there exist an induction functor $\mathrm{Ind}^d : \mathrm{Rep}\, S_n \to \mathrm{Rep}^d\, GL_n$ such that

$$\mathrm{Hom}_{S_n}(\mathrm{Res}\, W_\lambda, V_\mu) \cong \mathrm{Hom}_{GL_n}(W_\lambda, \mathrm{Ind}^d\, V_\mu)$$

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and $\pi^G(g)f(x) = f(xg)$.

Induced representation

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Given a representation (π, V) of H, $H \leq G$, can define $(\pi^G, \operatorname{Ind}_H^G V) \in \mathrm{Rep}\, G$

$\operatorname{Ind}_H^G V := \{f : G \to V \mid f(hg) = \pi(h)f(g), \forall h \in H, \forall g \in G\}$ and $\pi^G(g)f(x) = f(xg)$.

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Our work

We adapt Mackey's construction to the setting of polynomial representations.

The construction of polynomial induction

Notation: M_n: Ring of $n \times n$ matrices with entries in $\mathbf{C}$.
$P^d(M_n)$: Space of homogeneous polynomials of degree d in the entries of matrices $Q \in M_n$.

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Example

Let 1_n denote the trivial representation of S_n.

$$\mathrm{Ind}^d\, 1_n = \{f \in P^d(M_n) \mid f(wQ) = f(Q) \text{ for all } w \in S_n,\ Q \in M_n\}$$

Results on Polynomial Induction

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Corollary (Representation theorytic view of Littlewood's identity)

$r_{\lambda,\mu} = \langle \mathrm{Res} W_\lambda, V_\mu \rangle_{S_n} = \langle s_\lambda, s_\mu[1 + h_1 + h_2 + \cdots] \rangle.$

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Thank you for your attention!