Stable envelopes, 3d mirror symmetry, bow varieties

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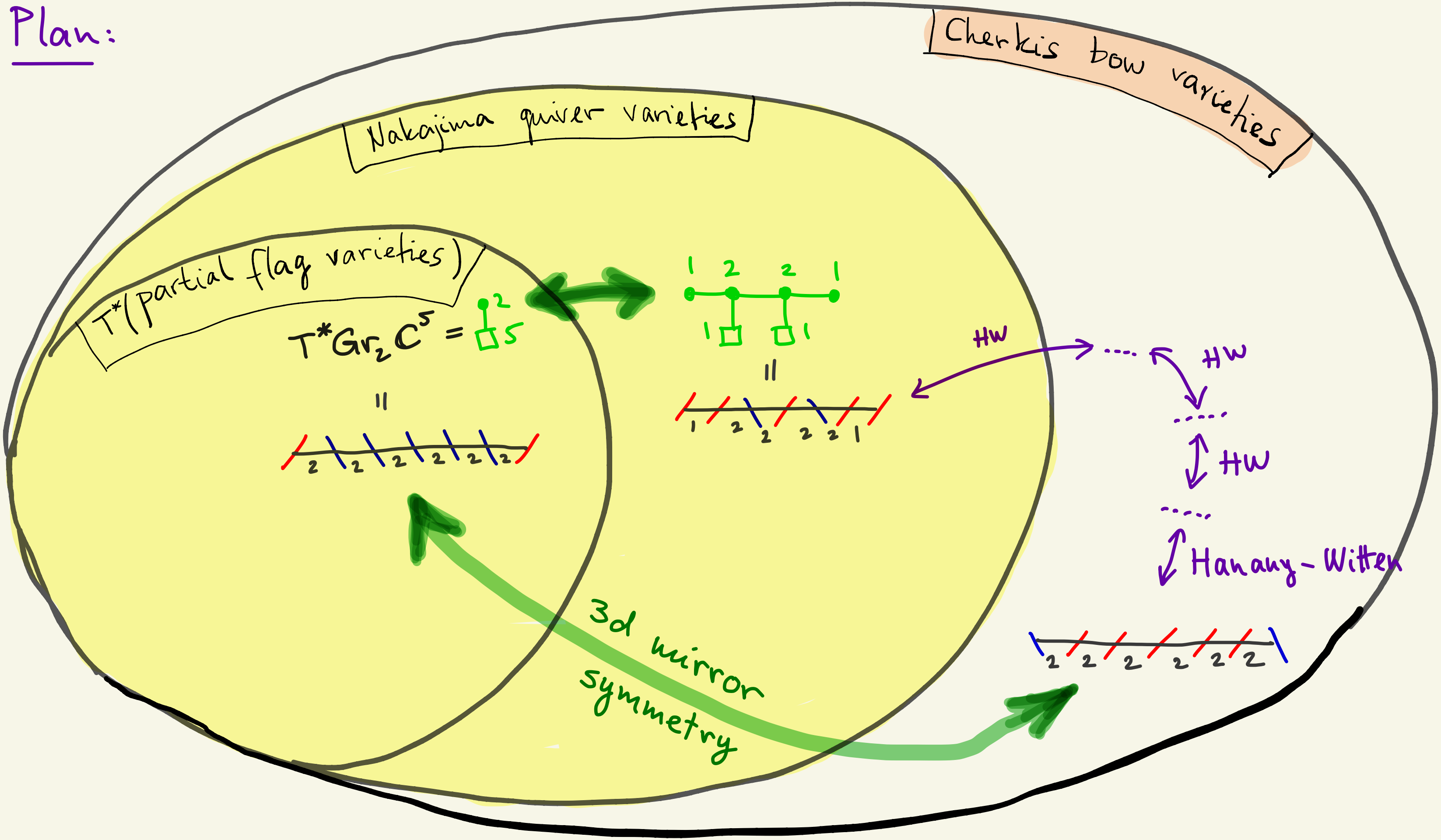
- joint work with Yiyan Shou

- learned about branes from Lev Rozansky

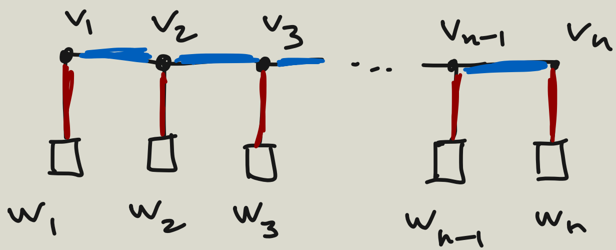
- related works with

  Andrey Smirnov
  Alexander Varchenko
  Zijun Zhou
  Andrzej Weber

Plan:


Cherkis bow varieties
Nakajima quiver varieties
T*(partial flag varieties)
$T^*Gr_2\mathbb{C}^5 = $ 2 5
=
2 2 2 2 2 2
1 2 2 1
1 1
=
1 2 2 2 2 1
HW
HW
HW
Hanany-Witten
2 2 2 2 2 2
3d mirror symmetry

Nakajima quiver varieties:



quiver Q

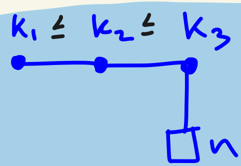
$\rightsquigarrow$ $\mathcal{N}(Q)$

quiver variety

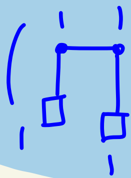
Ex

$\mathcal{N}\Big($ 

$\Big) = T^*\mathrm{Gr}_k\,\mathbb{C}^n$

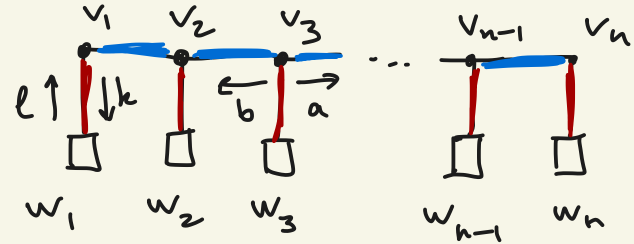
$\mathcal{N}\Big($ 

$\Big) = T^*\mathcal{F}_{k_1,k_2,k_3,n}$

$\mathcal{N}\Big($ 

$\Big) = \widetilde{\mathbb{C}^2/\mathbb{Z}_3}$

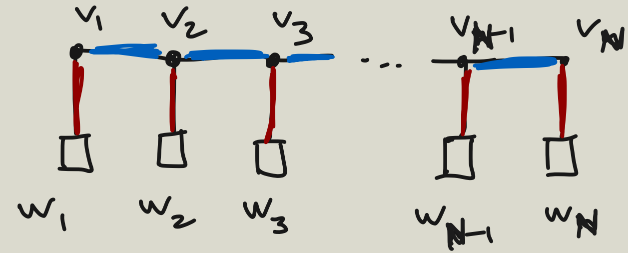
def

$\mathcal{N}\Bigg($ 

$\Bigg) = \ldots$

- $R := \bigoplus_i \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_{i+1}}) \oplus \bigoplus_i \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$
- $\mu : R \oplus R^* \longrightarrow \bigoplus_i \mathrm{End}(\mathbb{C}^{v_i})$ $\qquad \mu = [a, b] - \ell k$
- $N(Q) := \mu^{-1}(0)^{ss} \big/ \times_i GL_{v_i}$

$\mathcal{N}(Q)$ (type A)



- smooth
- holomorphic symplectic
- $T = \left(T^{w_1} \times T^{w_2} \times \dots \times T^{w_N}\right) \times \mathbb{C}^*_\hbar$ action
- finitely many fixed pts
- "tautological" v_1-, v_2-, $\dots$, v_N- bundles

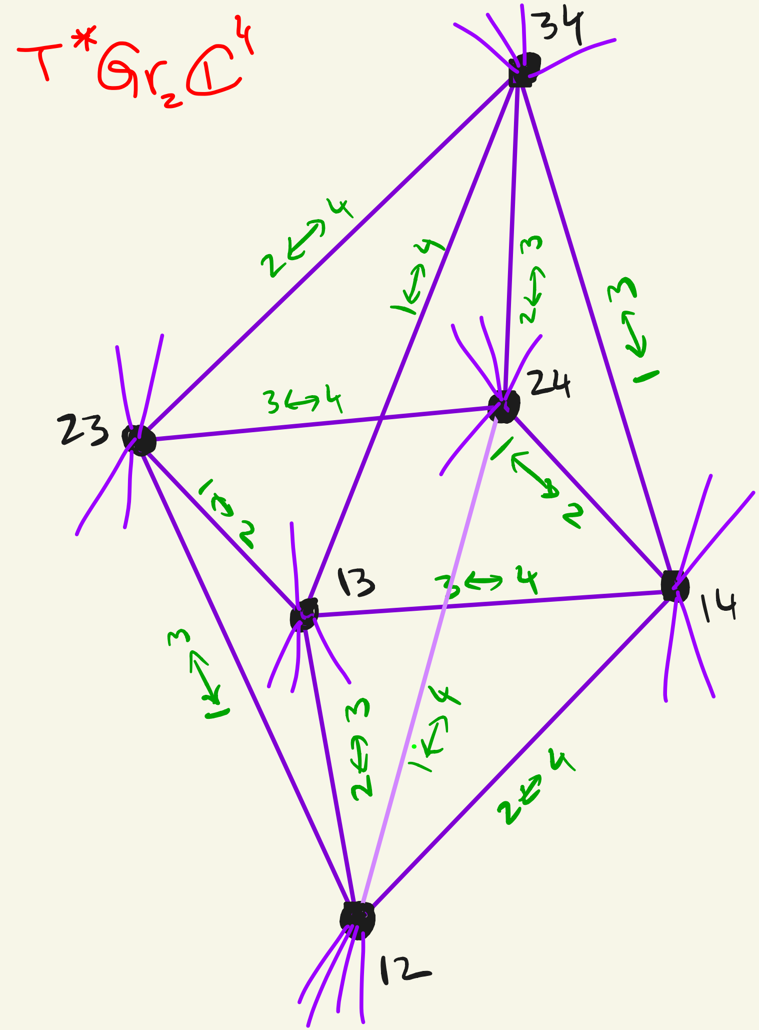
$$H_T^*(\mathcal{N}(Q)) = ?$$

$$H_T^*(\mathcal{N}(Q)) \hookrightarrow \bigoplus_{p\ T\text{-fix}} \underbrace{H_T^*(p)}_{\mathbb{C}[z_1,\dots,z_n,\hbar]}$$

Localization map, Loc

$$\mathrm{im}(\mathrm{Loc}) = ?$$

constraints among the components

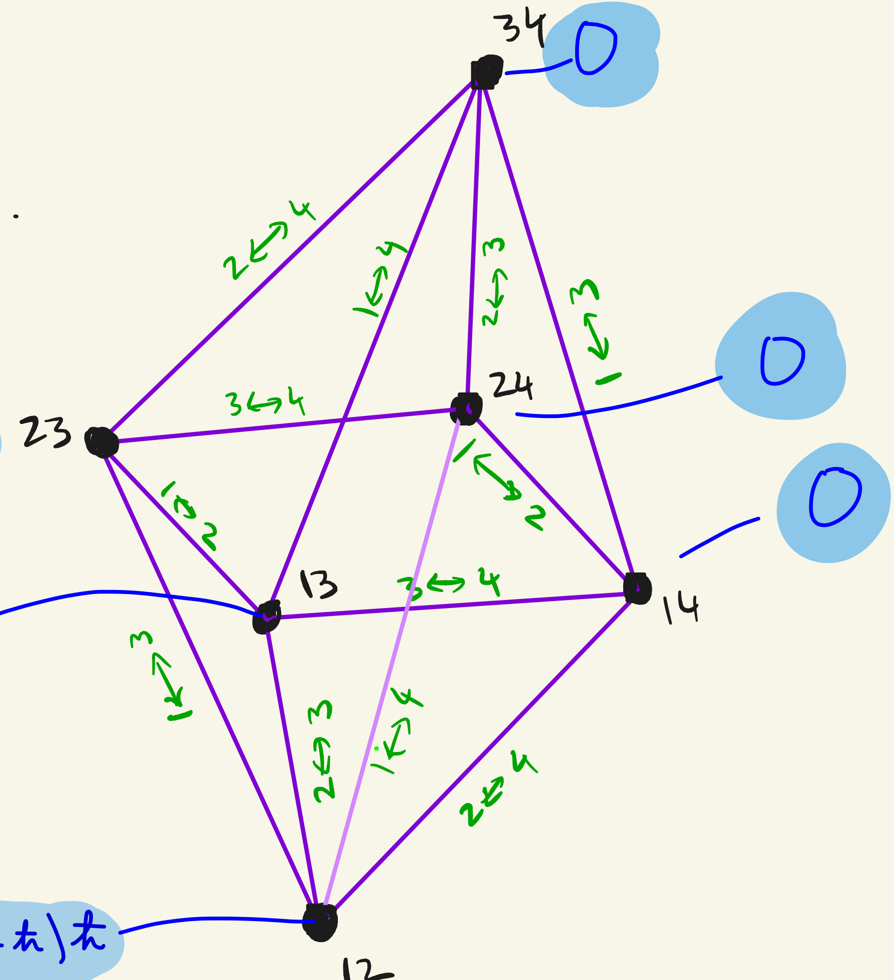


For example, this 6-tuple is an element of $H^*_T(T^*Gr_2\mathbb{C}^4)$.

$(z_4-z_3)(z_4-z_2)(z_2-z_1+\hbar)(z_3-z_1+\hbar)$

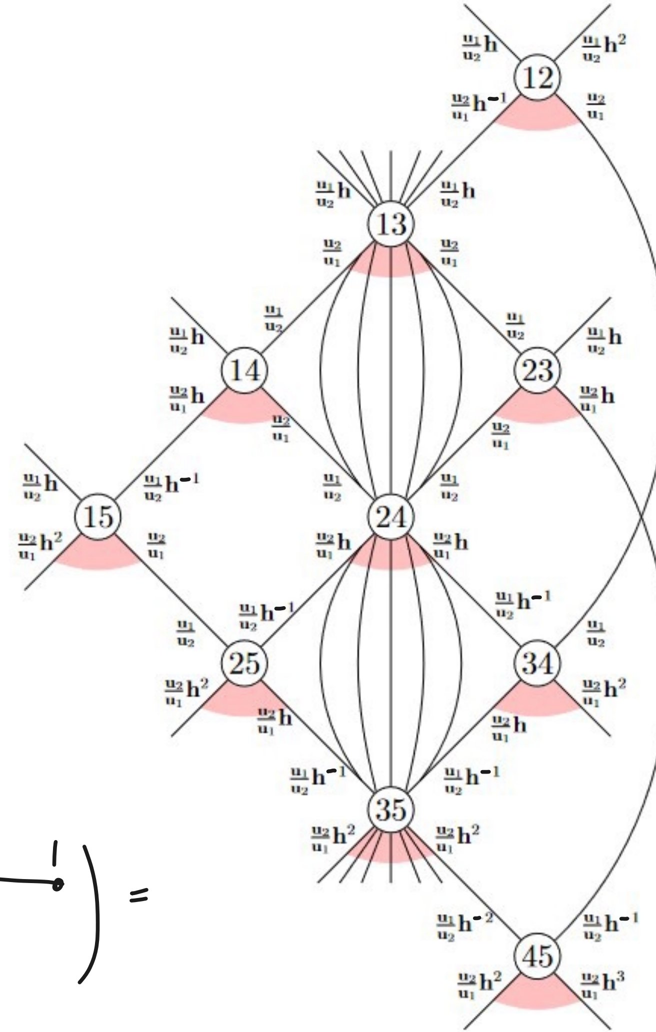
$(z_4-z_1)(z_4-z_3)(z_3-z_2+\hbar)\hbar$

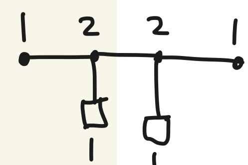
$(z_4-z_1)(z_4-z_2)(z_2-z_3+\hbar)\hbar$



Warning

- $T^*Gr_2\mathbb{C}^4$ was special ("GKM")
- In general the constraints among components are more restrictive



$\mathcal{N}\Big($  $\Big) =$

Towards $\mathrm{Stab}_p \in H_T^*(\mathcal{N}(Q))$

(torus fixed point)

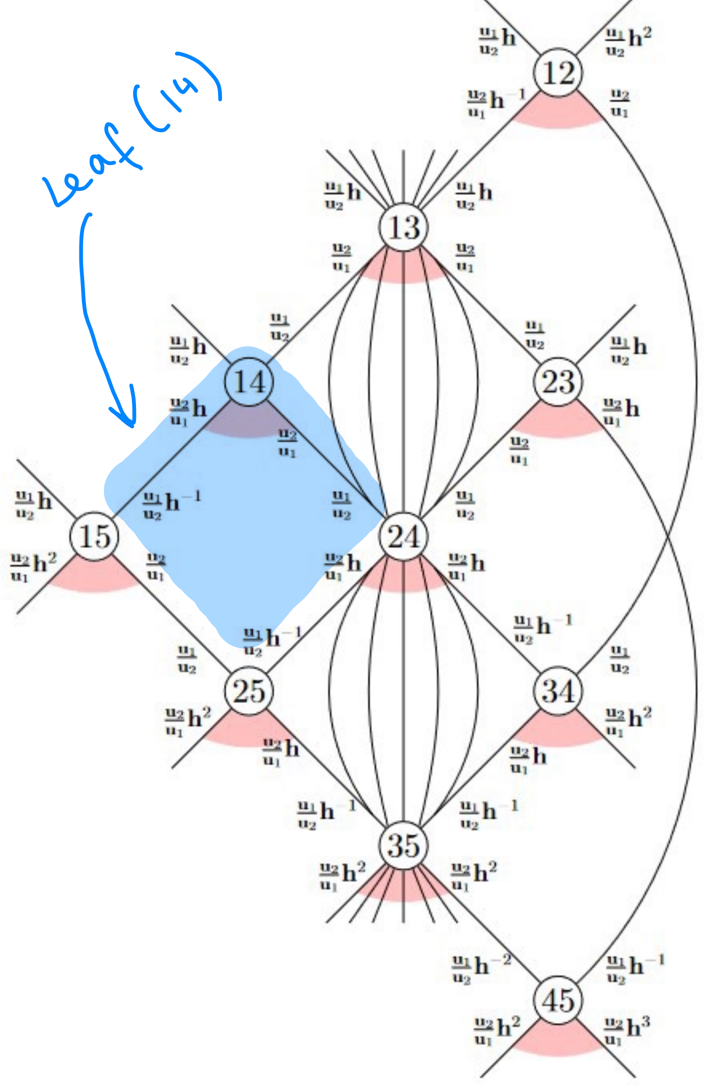
- fix $\mathbb{C}^* \xrightarrow{\sigma} T$

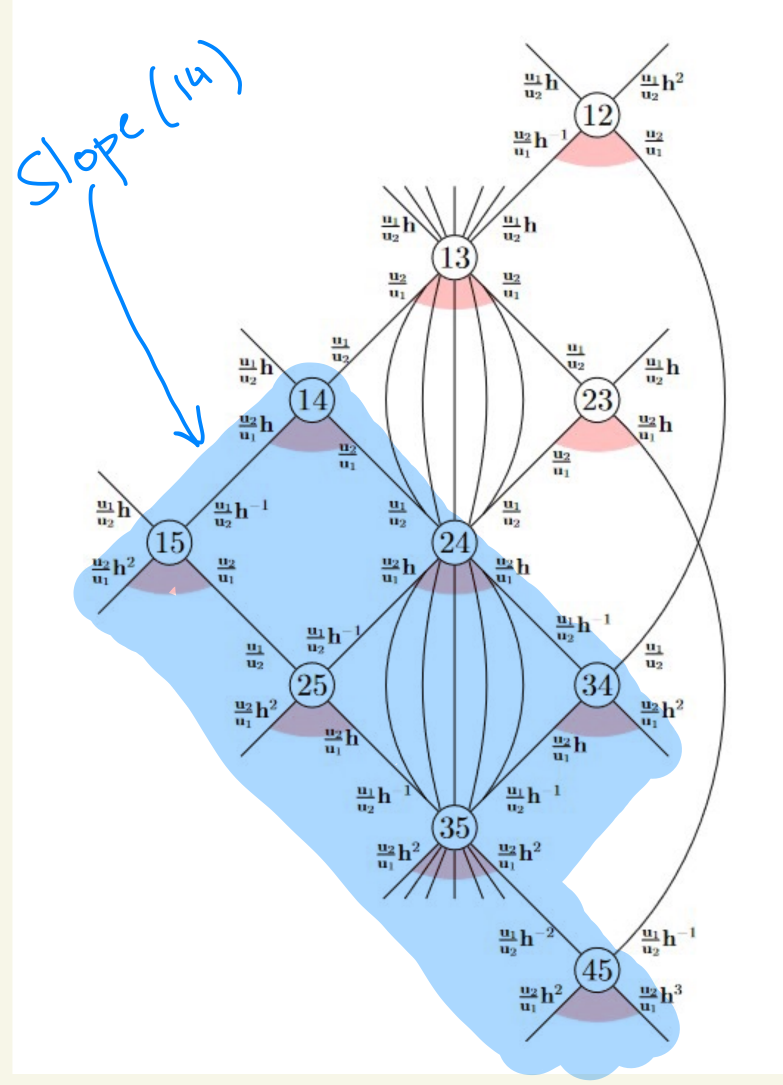
  $z \longmapsto (z, z^2, z^3, \dots, z^n, 1)$

- $p \in \mathcal{N}(Q)^T$ $\quad \mathrm{Leaf}(p) = \{ x \in \mathcal{N}(Q) : \lim_{z \to 0} \sigma(z) x = p \}$

- $p' \leq p$ if $\overline{\mathrm{Leaf}(p)} \ni p'$

- $\mathrm{Slope}(p) := \bigcup_{p' \leq p} \mathrm{Leaf}(p')$


Leaf (14)
12
13
14
23
15
24
25
34
35
45


Slope (14)
12
13
14
23
15
24
25
34
35
45

def [MO] $\mathrm{Stab}_p \in H^*_T(\mathcal{N}(Q))$ is the unique class

- support axiom:

  supported on $\mathrm{Slope}(p)$

- normalization axiom:

  $\mathrm{Stab}_p\big|_p = e\left(\nu\left(\mathrm{Slope}_p\right)\right)$

- boundary axiom:

  $\mathrm{Stab}_p\big|_q$ divisible by $\hbar$ for $p \neq q$

Stab_{14}

@ 12, 13, 23 $= 0$

@ 14 $= (z_1 - z_2)(z_1 - z_2 + \hbar)$

@ 15 divisible by $(z_1 - z_2 + \hbar)$
divisible by $\hbar$

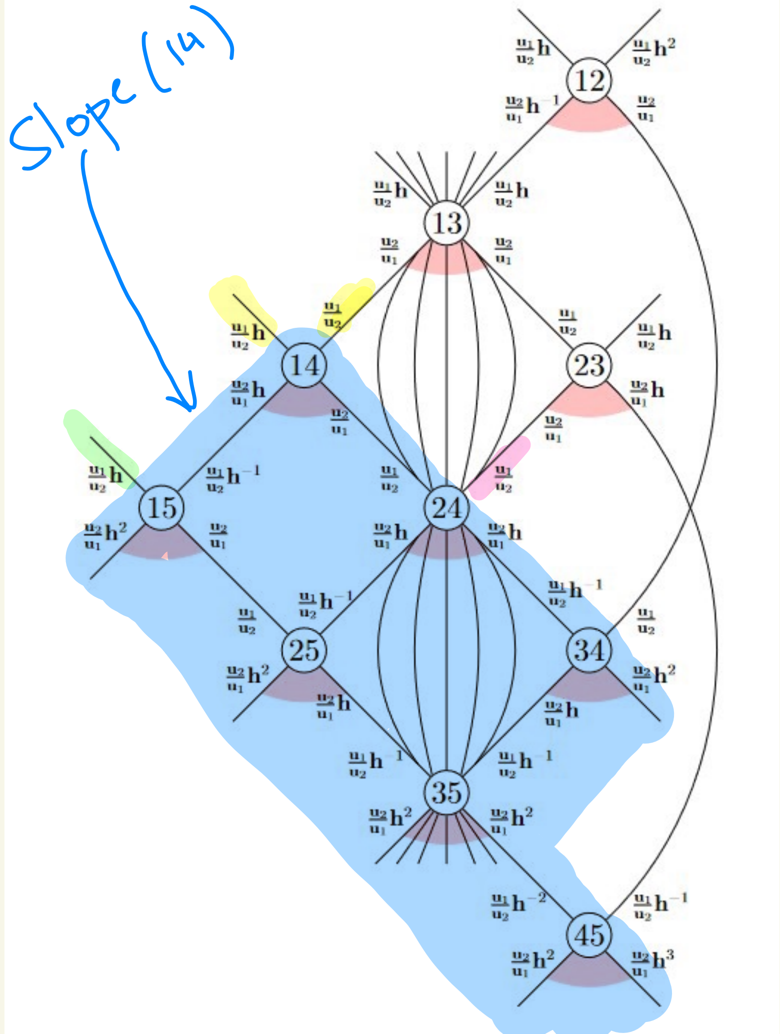
@ 24 divisible by $(z_1 - z_2)$
divisible by $\hbar$

@ 25 divisible by $\hbar$

@ 34 divisible by $(z_1 - z_2)$
divisible by $\hbar$

@ 35 divisible by $\hbar$

@ 45 divisible by $(z_1 - z_2 - \hbar)$
divisible by $\hbar$



REMARK (2 SLIDES) ON GEOM. REPR. THEORY:

Stab's define geometric R-matrices & quantum group actions. [MO]

- $\sigma = (z, z^2, z^3, \dots, z^n, 1)$ $\quad H^*_T(\mathcal{N}(Q)^T) \xrightarrow{\mathrm{Stab}_\sigma} H^*_T(\mathcal{N}(Q))$

  $$1_p \longmapsto \mathrm{Stab}_\sigma\, p$$

- other 1-parameter subgroups also define Stab's

  $$H^*_T(\mathcal{N}(Q)^T) \underset{\mathrm{Stab}_{\sigma'}}{\overset{\mathrm{Stab}_\sigma}{\rightrightarrows}} H^*(\mathcal{N}(Q)) \otimes \underline{\mathbb{C}(z, \hbar)}$$

- $\mathrm{Stab}_\sigma^{-1} \circ \mathrm{Stab}_{\sigma'} =:$ "geometric R-matrix"

$$\mathcal{N} := T^*Gr_0\mathbb{C}^2 \sqcup T^*Gr_1\mathbb{C}^2 \sqcup T^*Gr_2\mathbb{C}^2$$

$$\sigma = (z_1, z_2, 1)$$
$$\sigma' = (z_2, z_1, 1)$$

$$H_T^*(\mathcal{N}^T) \underset{\mathrm{Stab}_{\sigma'}}{\overset{\mathrm{Stab}_\sigma}{\rightrightarrows}} H_T^*(\mathcal{N})$$

$T^*Gr_0\mathbb{C}^2$	$1 \longmapsto$	1	1
$T^*Gr_1\mathbb{C}^2$	$1_{10} \longmapsto$	$(z_2 - z_1,\ 0)$	$(z_2 - z_1 + \hbar,\ \hbar)$
	$1_{01} \longmapsto$	$(\hbar,\ z_1 - z_2 + \hbar)$	$(0,\ z_1 - z_2)$
$T^*Gr_2\mathbb{C}^2$	$1 \longmapsto$	1	1

$$\mathrm{Stab}_\sigma^{-1} \circ \mathrm{Stab}_{\sigma'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & \frac{\hbar}{z_1 - z_2 + \hbar} & 0 \\ 0 & \frac{\hbar}{z_1 - z_2 + \hbar} & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

END OF REMARK

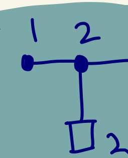
So far :

$\mathrm{Stab}_p \in H_T^*(\mathcal{N}(Q))$

p: T-fixed point of $\mathcal{N}(Q)$

- defined axiomatically
- remark : main ingredients of defining quantum group actions on $H_T^*(\mathcal{N}(Q))$.

THE COINCIDENCE !!!

$T^*\mathrm{Gr}_2\mathbb{C}^4 \longleftrightarrow \mathcal{N}\Big($  $\Big)$

[RSVZ 2020]

intimate relationship between their Stable Envelopes

dim = 8

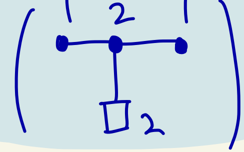
fix pts = 6

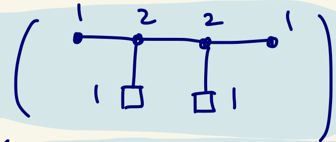
$T^4 \times \mathbb{C}^*_\hbar$ action

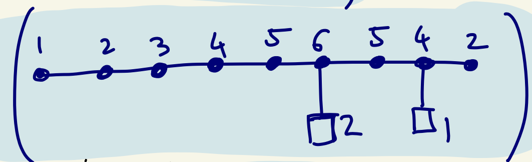
dim = 4

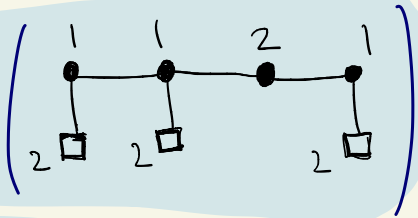
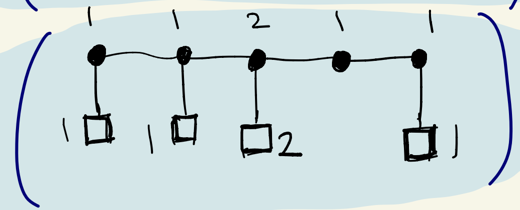
fix pts = 6

$T^2 \times \mathbb{C}^*_\hbar$ action

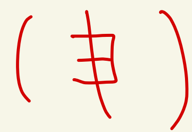
8 $T^*Gr_2\mathbb{C}^4$ $\longleftrightarrow$ $\mathcal{N}$  4 [RSVZ]

12 $T^*Gr_2\mathbb{C}^5$ $\longleftrightarrow$ $\mathcal{N}$  4

64 $T^*\mathcal{F}_{2,6,10}$ $\longleftrightarrow$ $\mathcal{N}$  16 [RS]

8 $\mathcal{N}$  $\longleftrightarrow$ $\mathcal{N}$  10

T^*G/B $\longleftrightarrow$ T^*G^L/B^L [RW 2020]

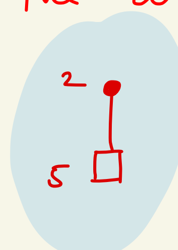
32 $T^*\mathcal{F}_{2,5,7}$ $\longleftrightarrow$ $\mathcal{N}$ 

$\boxed{\text{dim}}$

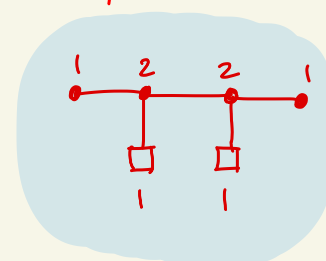
1. What exactly is the relationship between Stable Envelopes of 3d mirror dual spaces?

2. How to find the 3d mirror dual?

(ie what is the combinatorics that connects

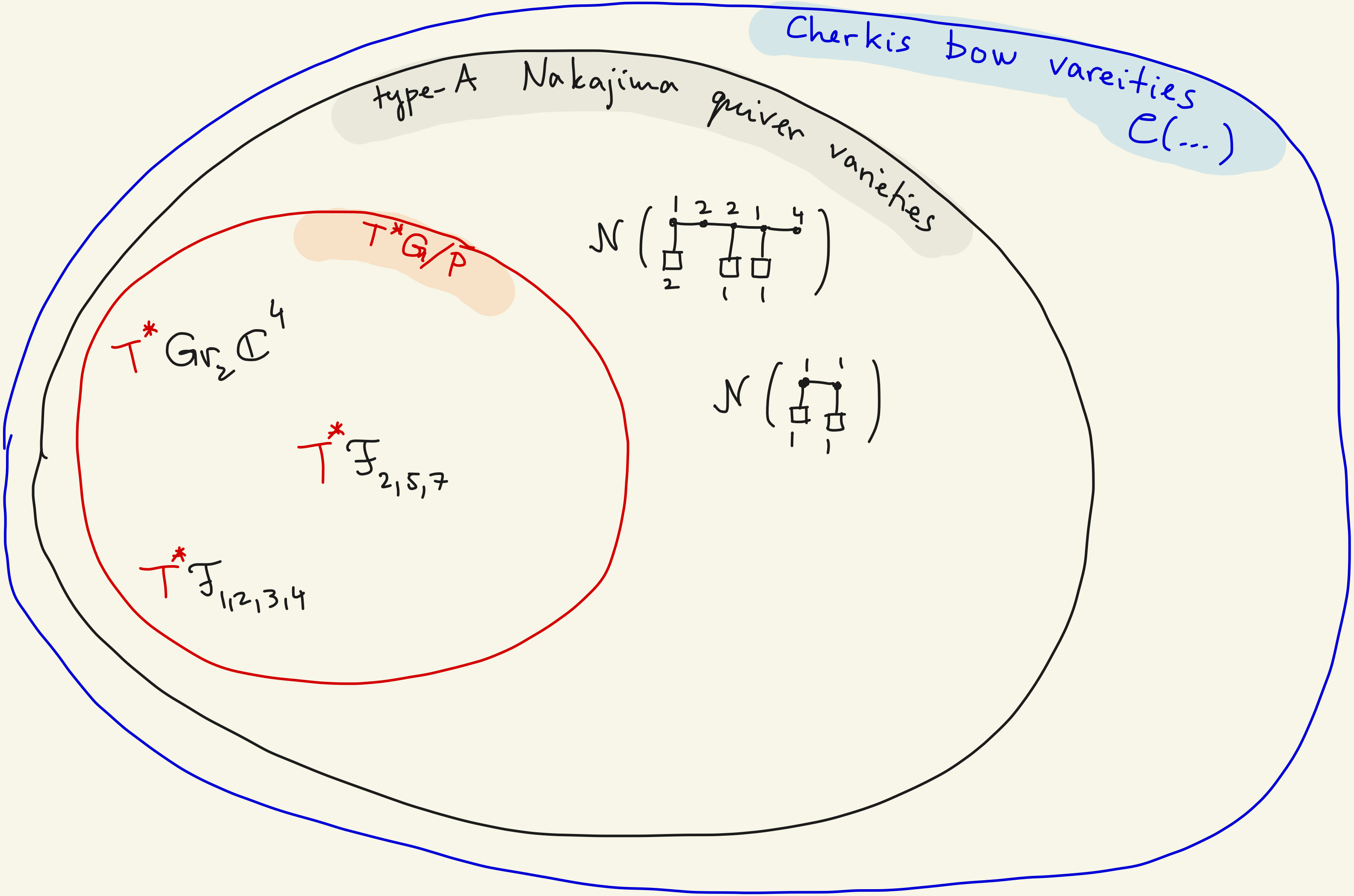


with

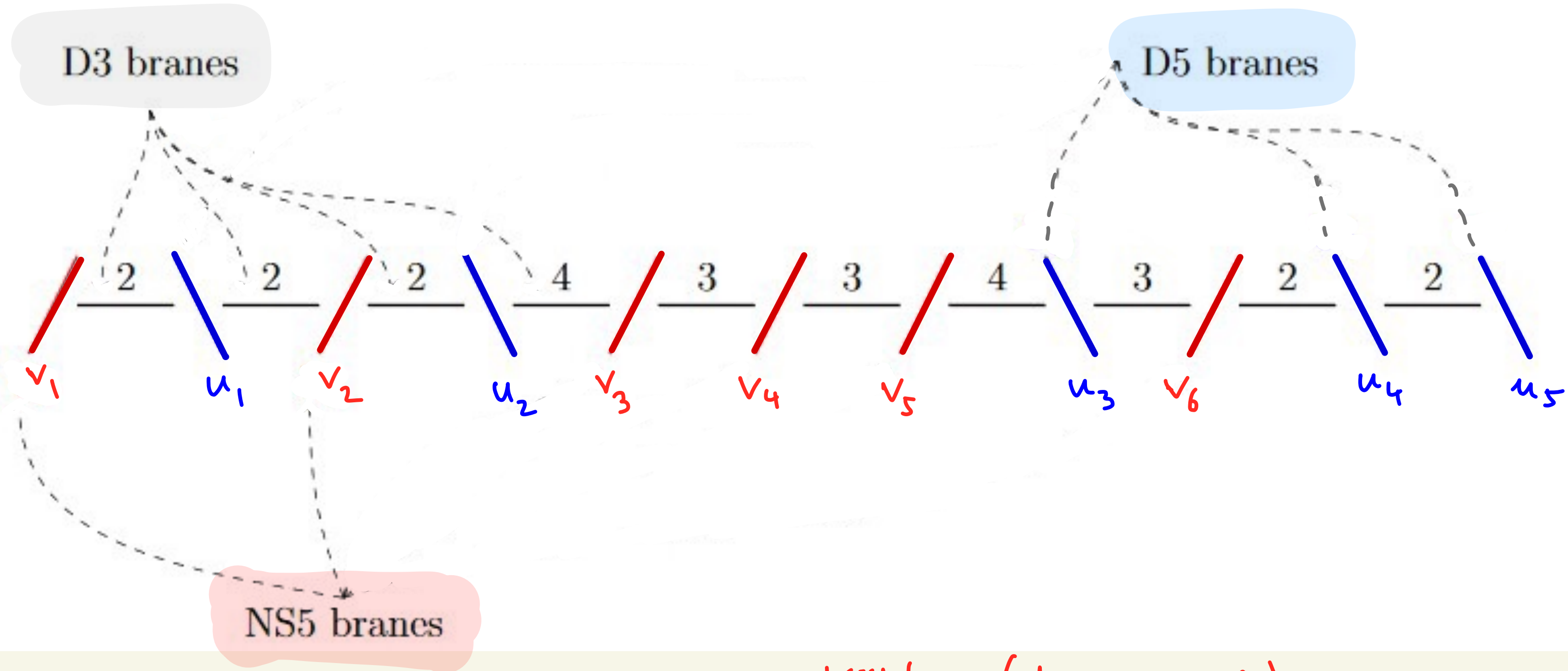


?)

(what is the mirror of $T^*\mathcal{F}_{2,5,7}$?)


Cherkis bow vareities
$\mathcal{C}(\dots)$
type-A Nakajima quiver varieties
T^*G/P
$T^*Gr_2\mathbb{C}^4$
$T^*\mathcal{F}_{2,5,7}$
$T^*\mathcal{F}_{1,2,3,4}$
$\mathcal{N}(\ )$
1 2 2 1 4
2 1 1
$\mathcal{N}(\ )$
1 1
1 1

Brane diagrams



v_i : Kähler (dynamical) variables

u_i : equivariant variables

tautological bundles,
one for each D3 brane

brane diagram $\mathcal{D}$ $\longrightarrow$ $C(\mathcal{D})$ $\curvearrowleft$ T D5 branes

Cherkis bow variety

Cherkis:
moduli space of unitary instantons on multi-Taub-NUT spaces
(key: Nahm's equation)

Nakajima-Takayama
Hamiltonian reduction of representations of certain quivers with relations

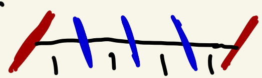
$\sim$

Rozansky-R
"symplectic intersection" of generalized Lagrange varieties

$$\dim\big(\mathcal{C}(D)\big) = \sum_{U\ D5}\Big[(d_{u_-}+1)d_{u_-} + (d_{u_+}+1)d_{u^+}\Big]$$

$$+ \sum_{V\ NS5} 2\, d_{v^+} d_{v^-} \quad - 2\sum_{X\ D3} d_X^2$$

example

$\dim\Big(\mathcal{C}\big($  $\big)\Big) = 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1$

$+ 2\cdot 0\cdot 1 + 2\cdot 1\cdot 0 \quad - 2\left(1^2+1^2+1^2+1^2\right)$

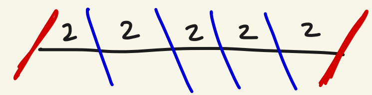
$= 4$

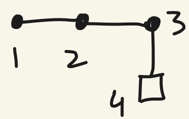
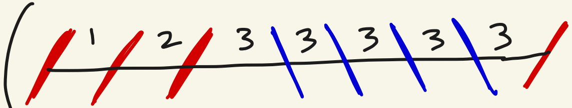
$T^*\mathbb{P}^2$

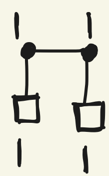
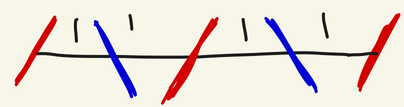
How are $\mathcal{N}$(quiver) special cases?

<u>Examples</u>

$T^*\mathbb{P}^1 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

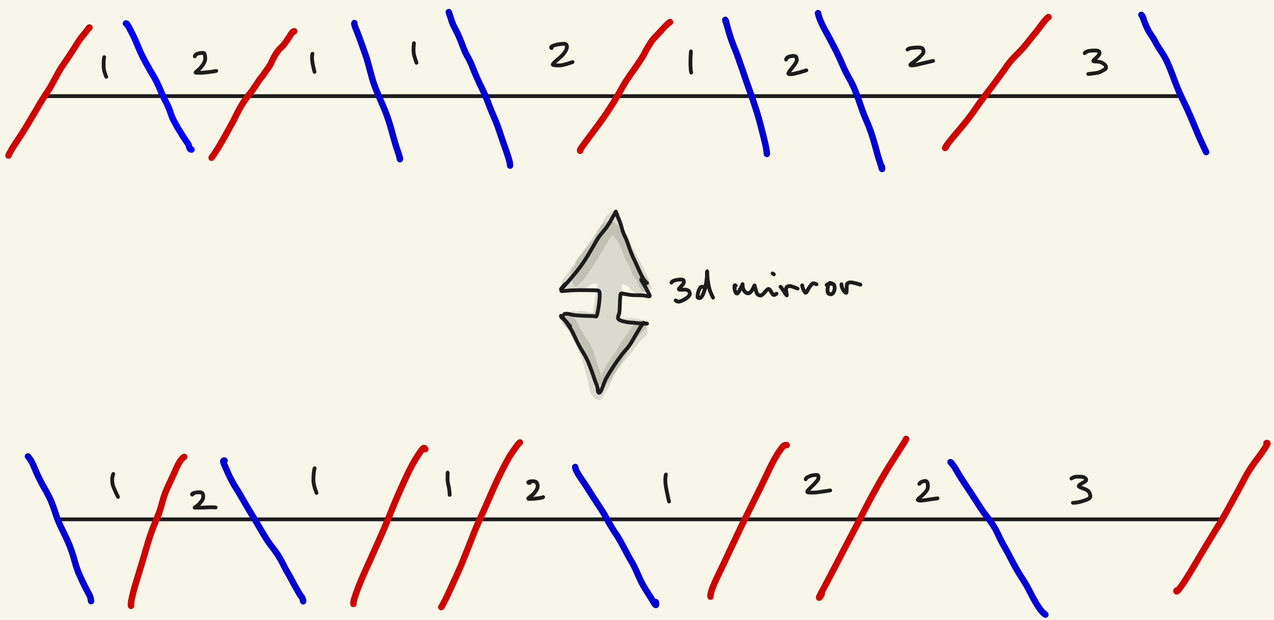
$T^*\mathrm{Gr}_2\mathbb{C}^4 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

$T^*\mathcal{F}_{1,2,3,4} = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

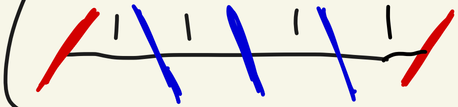
$\mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

<u>Observe</u>  "cobalanced brane diagram"

3D mirror symmetry for bow varieties:



Ex

$T^*\mathbb{P}^2 = \mathcal{N}\left(\begin{smallmatrix}1 \\ | \\ \boxed{}_3\end{smallmatrix}\right) = \mathcal{C}\Bigg($ 

$\Bigg)$ dim 4

3d mirror

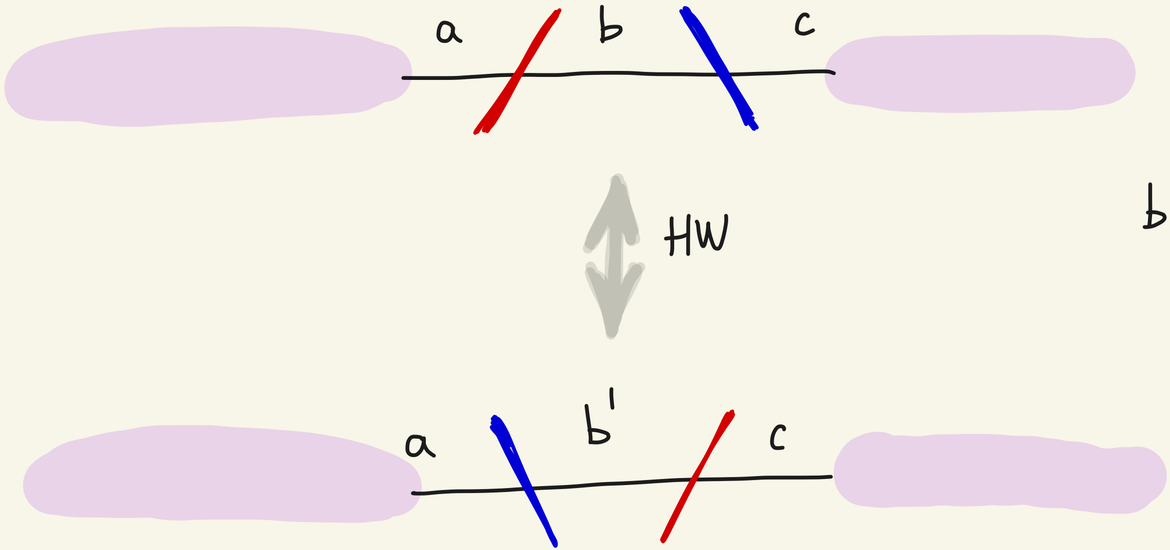
$\mathcal{C}\Bigg($ 

$\Bigg)$ dim 2

not cobalanced, ie not $\mathcal{N}(\dots)$

... but ... <to be continued>

Hanany-Witten transition on brane diagrams.

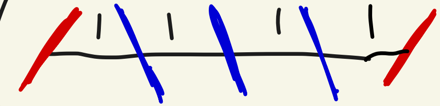


$$b + b' = a + c + 1$$

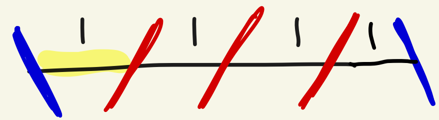
(why? later: "brane charge")

<u>Thm</u> $\mathcal{C}(\mathcal{D}) \cong \mathcal{C}(HW(\mathcal{D}))$

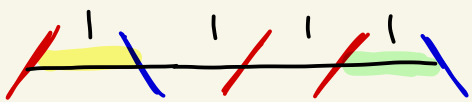
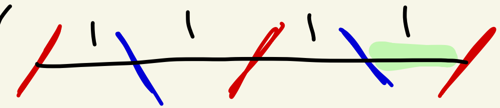
Ex

$T^*\mathbb{P}^2 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

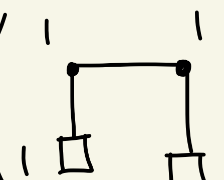
3d mirror

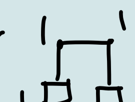
$\mathcal{C}\Big($  $\Big)$

HW

$\mathcal{C}\Big($  $\Big) \overset{HW}{=} \mathcal{C}\Big($  $\Big)$

$\|$

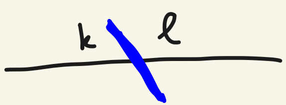
$\mathcal{N}\Big($  $\Big)$

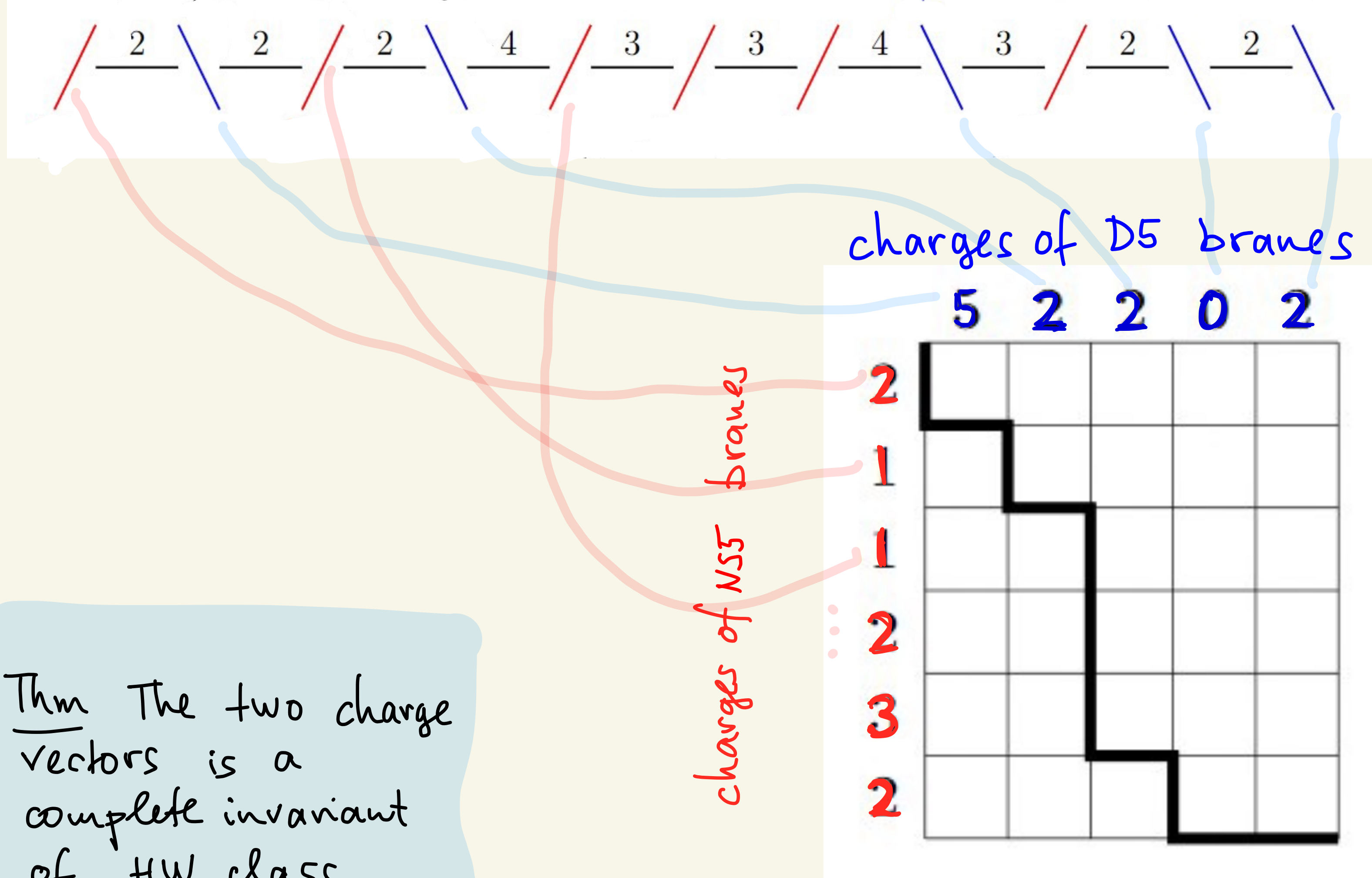
$\Rightarrow$ $T^*\mathbb{P}^2 \xleftrightarrow{\text{3d mirror}} \mathcal{N}\Big($  $\Big)$

def brane charge

$$\mathrm{charge}\left(\text{NS5 brane}\right) := \ell - k + \#\{\text{D5-branes left of it}\}$$

$$\mathrm{charge}\left(\text{D5 brane}\right) := k - \ell + \#\{\text{NS5-branes right of it}\}$$

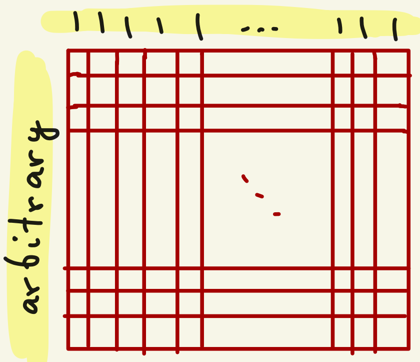


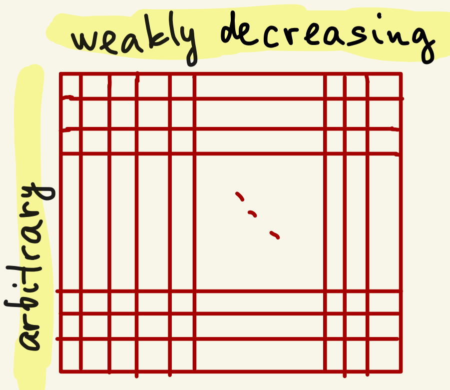


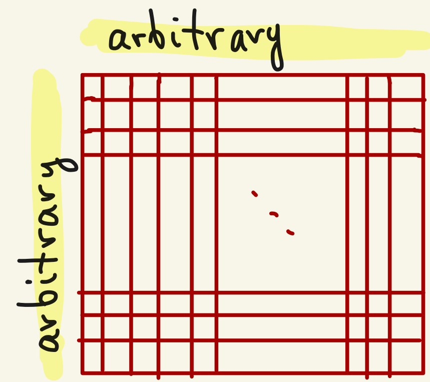
<u>Thm</u> The two charge vectors is a complete invariant of HW class

Thm (up to HW transitions)

$T^*G/P \quad \subset \quad \mathcal{N}(\text{quiver}) \quad \subset \quad \mathcal{C}(\text{brane diagram})$





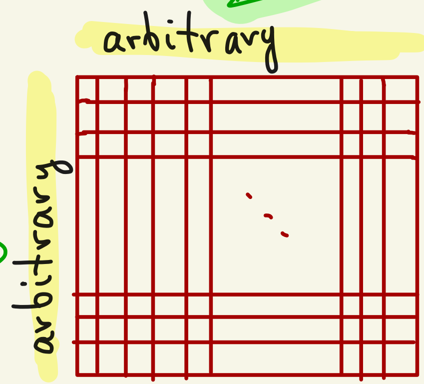


closed for transpose!

REMARK ON GEOM. REPR. TH. CONTINUED (1 SLIDE)

Expectation:
(known for quivers)

② which representation



① size: which quantum group

③ which weight space of the representation

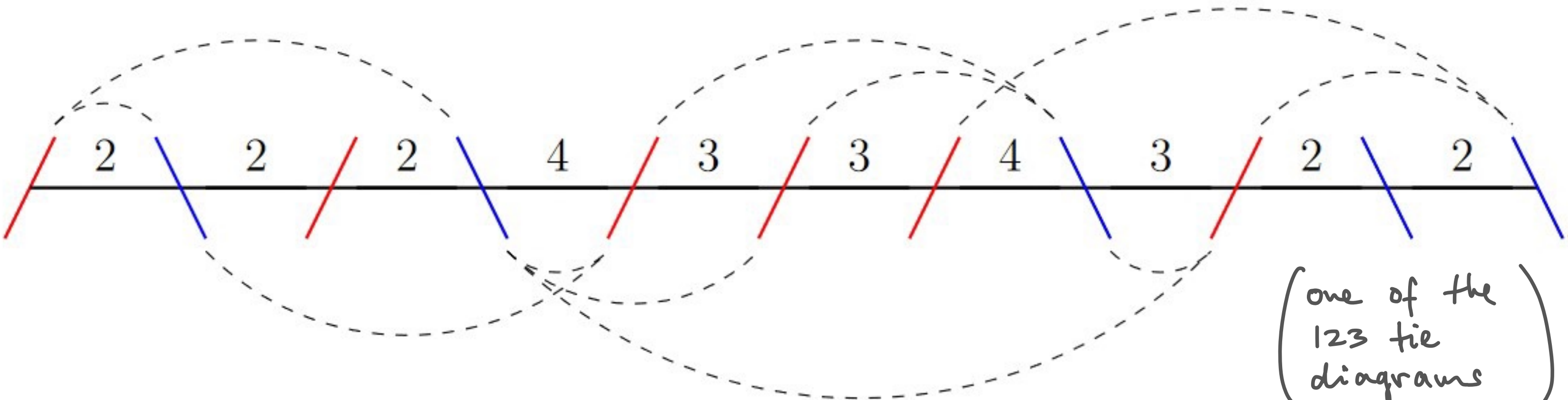
The first data we need for

- $H_T^*(\mathcal{C}(D))$
- stable envelopes:

torus fixed points

(a.k.a. "exact vacuums")

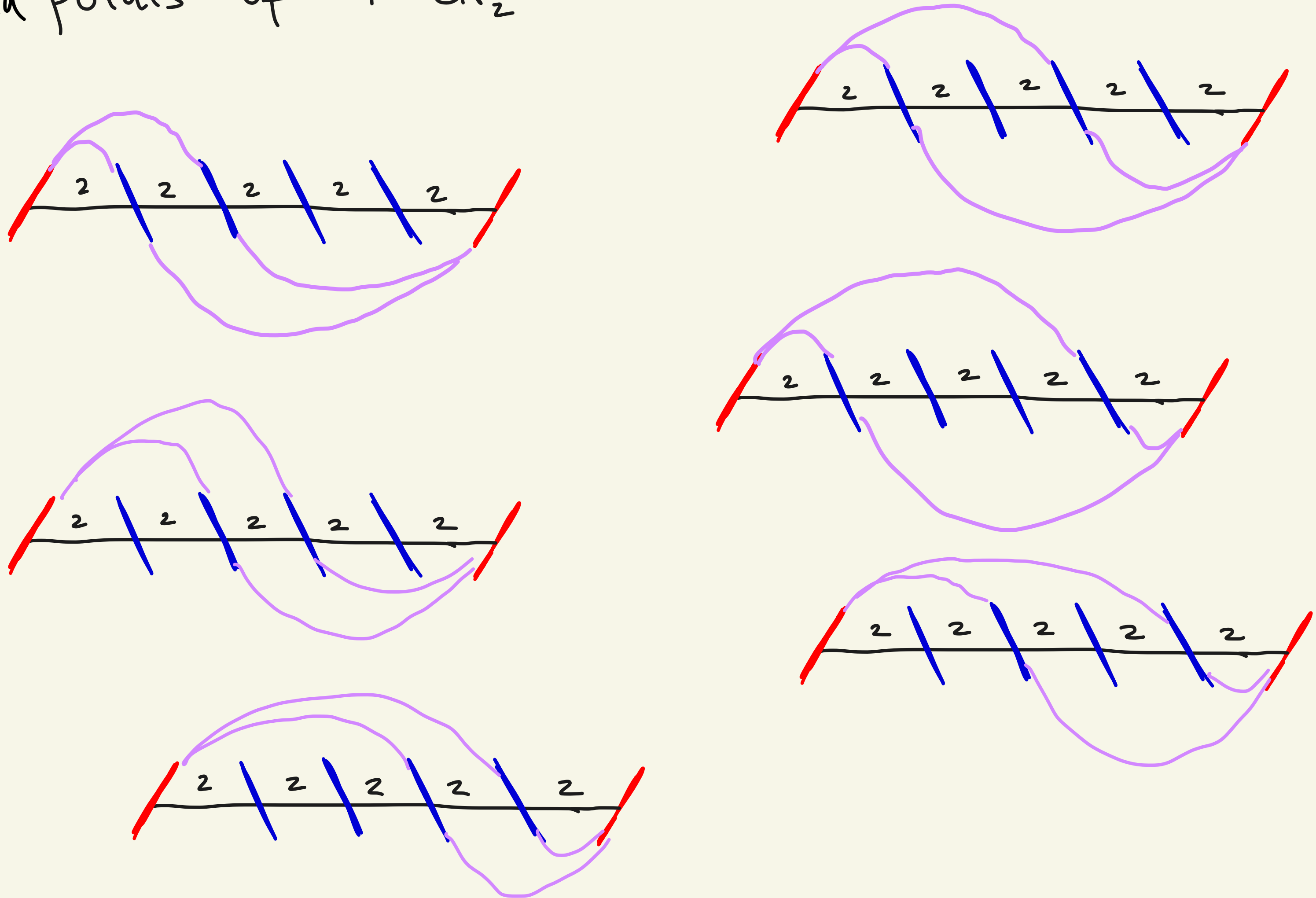
AirSpeed EXACT Reach AS3008A Upright Vacuum, Bagless, Allergy Filter, Blue/Black fixed points $\overset{1:1}{\longleftrightarrow}$ tie diagrams

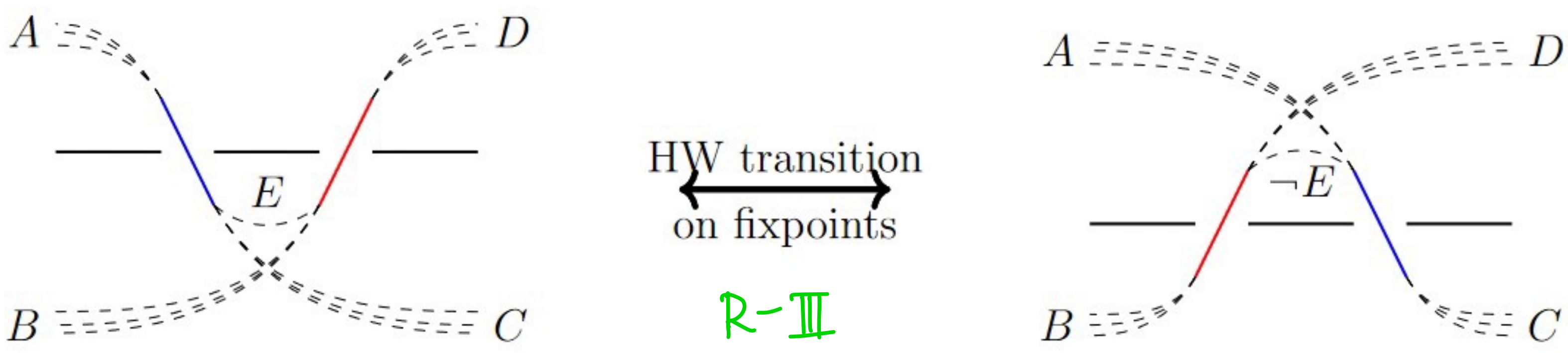


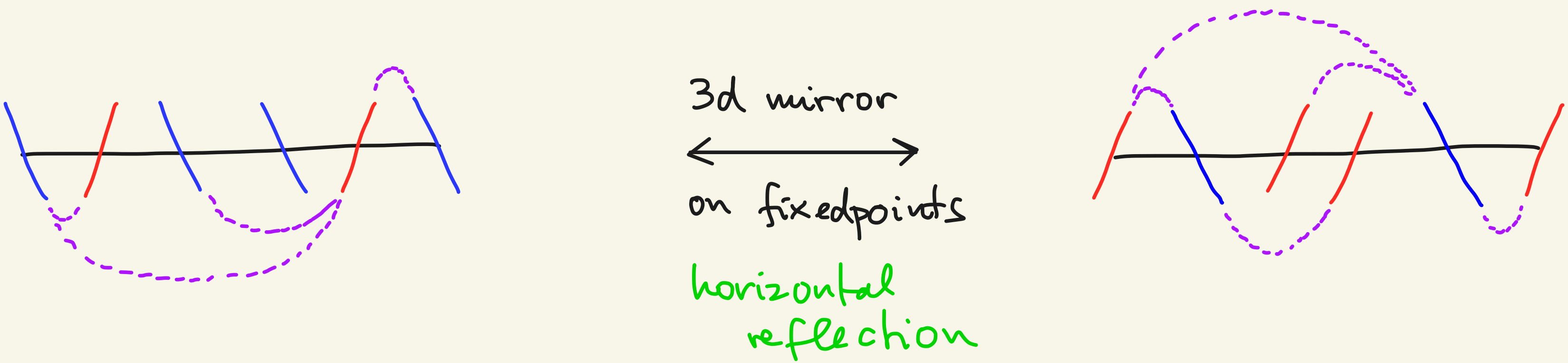
(one of the 123 tie diagrams)

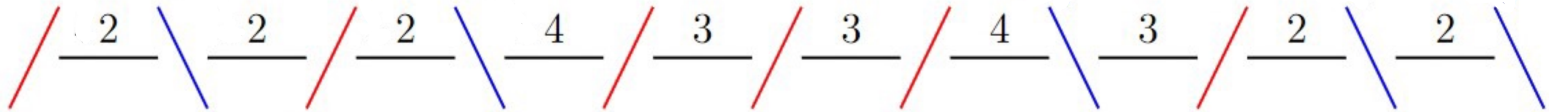
- a tie must connect 5-branes of different kinds
- each D3 brane to be cored as many times as its multiplicity

fixed points of $T^* Gr_2 \mathbb{C}^4$:




A
D
E
B
C
HW transition
on fixpoints
R-III
A
D
¬E
B
C


3d mirror
on fixedpoints
horizontal
reflection



binary contingency tables

BCT : 0-1-matrix with row & column sums the charge vectors

one of the 123 BCTs

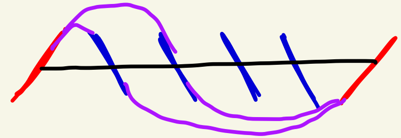
	5	2	2	0	2
2	1	1	0	0	0
1	1	0	0	0	0
1	0	0	1	0	0
2	1	0	1	0	0
3	1	1	0	0	1
2	1	0	0	0	1

Thm

fix pts $\overset{1:1}{\longleftrightarrow}$ BCT's

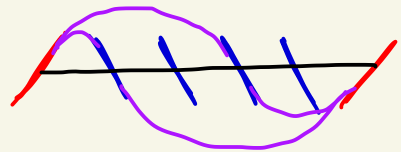
$Gr_2 \mathbb{C}^4$

$\{1,2\}$

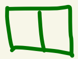


	1	1	1	1
2	1	1	0	0
2	0	0	1	1

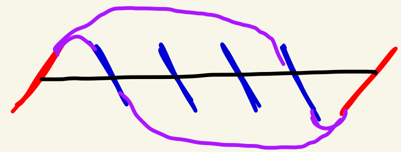
$\{1,3\}$



	1	1	1	1
2	1	0	1	0
2	0	1	0	1

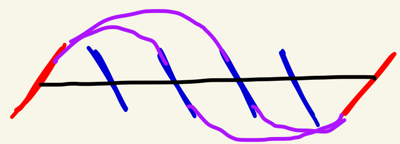


$\{1,4\}$

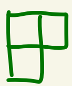


	1	1	1	1
2	1	0	0	1
2	0	1	1	0

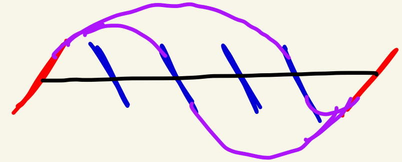
$\{2,3\}$



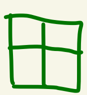
	1	1	1	1
2	0	1	1	0
2	1	0	0	1



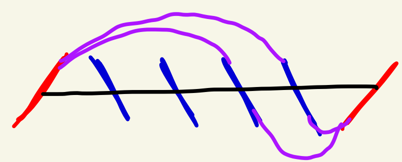
$\{2,4\}$



	1	1	1	1
2	0	1	0	1
2	1	0	1	0

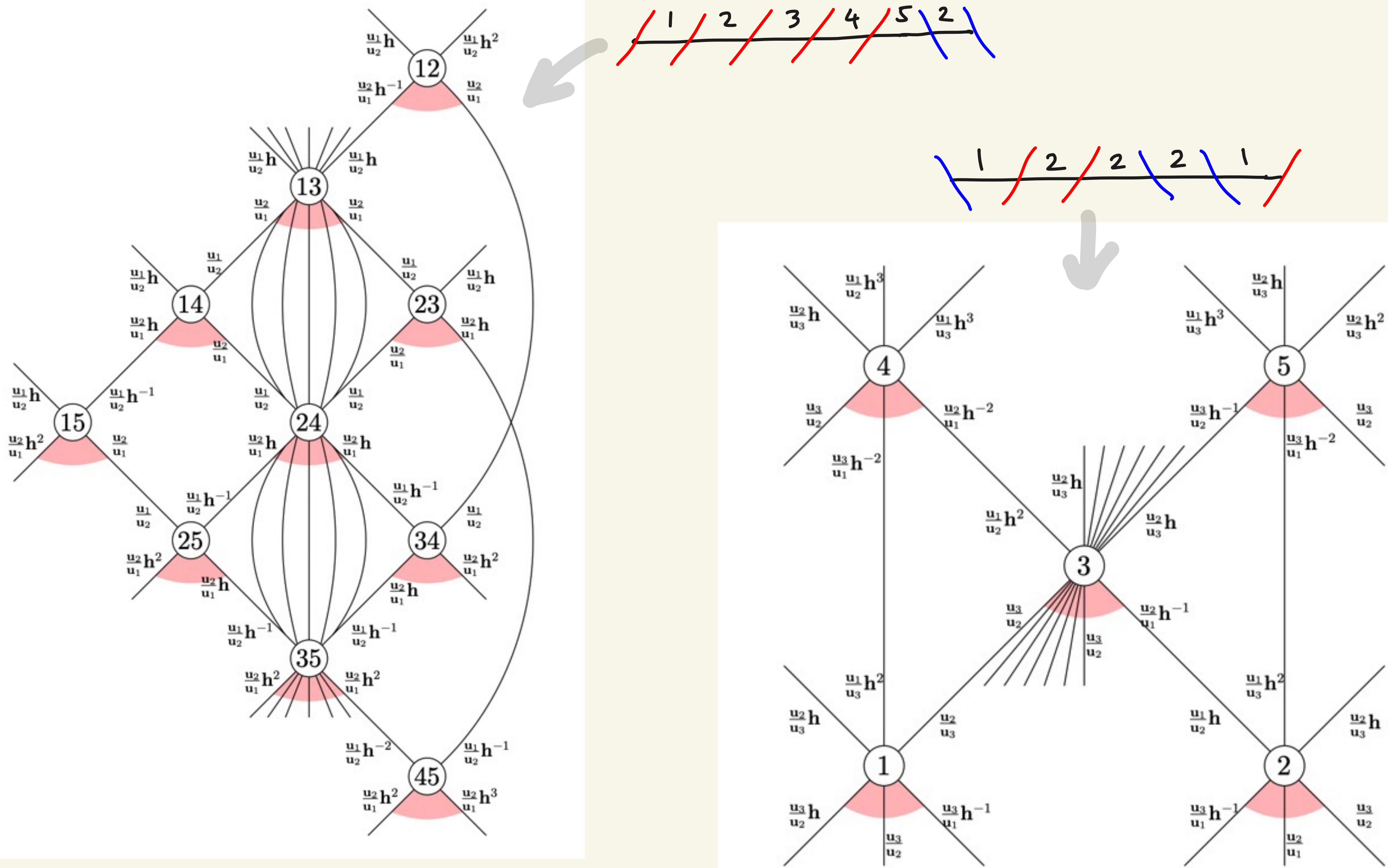


$\{3,4\}$

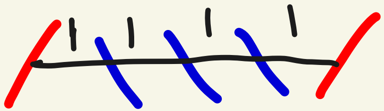


	1	1	1	1
2	0	0	1	1
2	1	1	0	0

fixed points
invariant curves (with weight) $\rightarrow$ moment graph $\rightarrow$ Stab_p

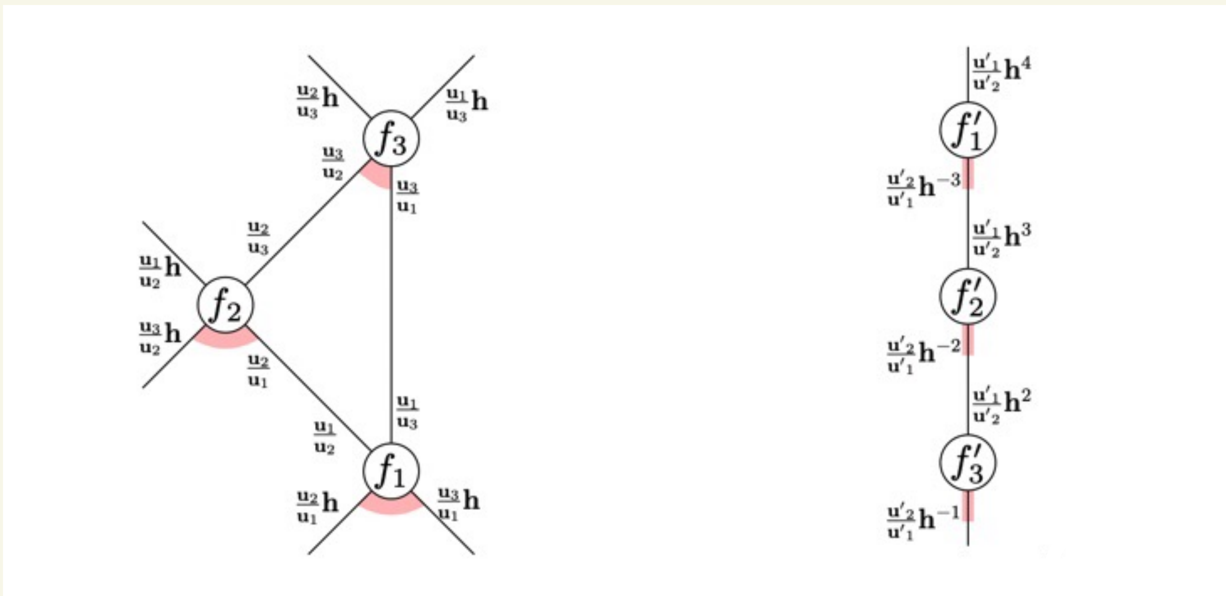


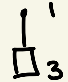
1 2 3 4 5 2
1 2 2 2 1
12
13
14
23
15
24
25
34
35
45
4
5
3
1
2

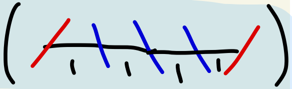


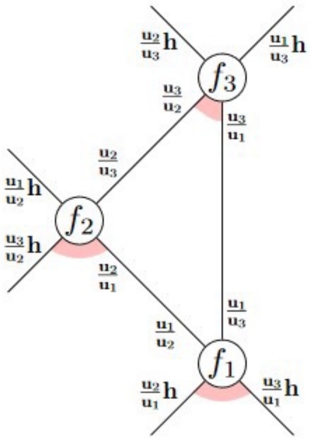
$T^*\mathbb{P}^2$

3d mirror$(T^*\mathbb{P}^2)$



$T^*\mathbb{P}^2 = \mathcal{N}\Big($  $\Big)$

$= \mathcal{C}\Big($  $\Big)$



	f_1	f_2	f_3
f_1	$\theta(\frac{u_1}{u_2})\theta(\frac{u_1}{u_3})\theta(\frac{v_2}{v_1}h^4)$	0	0
f_2	$\theta(h)\theta(\frac{u_1}{u_3})\theta(\frac{u_2 v_2}{u_1 v_1}h^3)$	$\theta(\frac{u_1}{u_2}h)\theta(\frac{u_2}{u_3})\theta(\frac{v_2}{v_1}h^3)$	0
f_3	$\theta(h)\theta(\frac{u_2}{u_1}h)\theta(\frac{u_3 v_2}{u_1 v_1}h^2)$	$\theta(h)\theta(\frac{u_1}{u_2}h)\theta(\frac{u_3 v_2}{u_2 v_1}h^2)$	$\theta(\frac{u_2}{u_3}h)\theta(\frac{u_1}{u_3}h)\theta(\frac{v_2}{v_1}h^2)$

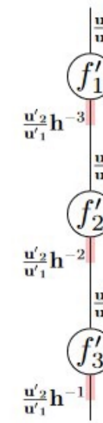
<u>Fact</u>: same after

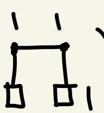
$\pm(\ )^T$

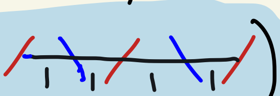
$u_\cdot \longleftrightarrow v_\cdot$

$h \longleftrightarrow h^{-1}$

In general: conjecture



$\mathcal{N}\Big($  $\Big) =$

$\mathcal{C}\Big($  $\Big)$

	f'_1	f'_2	f'_3
f'_1	$\theta(\frac{u'_1}{u'_2}h^4)\theta(\frac{v'_2}{v'_1})\theta(\frac{v'_3}{v'_1})$	$\theta(h)\theta(\frac{v'_3}{v'_1})\theta(\frac{v'_2 u'_2}{v'_1 u'_1}h^{-3})$	$\theta(h)\theta(\frac{v'_2}{v'_1}h^{-1})\theta(\frac{v'_3 u'_2}{v'_1 u'_1}h^{-2})$
f'_2	0	$\theta(\frac{u'_1}{u'_2}h^3)\theta(\frac{v'_2}{v'_1}h)\theta(\frac{v'_3}{v'_2})$	$\theta(h)\theta(\frac{v'_2}{v'_1}h)\theta(\frac{v'_3 u'_2}{v'_2 u'_1}h^{-2})$
f'_3	0	0	$\theta(\frac{u'_1}{u'_2}h^2)\theta(\frac{v'_3}{v'_2}h)\theta(\frac{v'_3}{v'_1}h)$

$$\left.\begin{array}{l} x_1+x_2+x_3=0 \\ y_1+y_2+y_3=0 \end{array}\right\} \Rightarrow \frac{1}{x_1x_2}+\frac{1}{x_2x_3}+\frac{1}{x_3x_1}=\frac{1}{y_1y_2}+\frac{1}{y_2y_3}+\frac{1}{y_3y_1}$$

H_T^*

$\uparrow$ rational limit $(\sin x \sim x)$

$x_1+x_2+x_3=0, \quad y_1+y_2+y_3=0 \Rightarrow$

$$\cot(x_1)\cot(x_2)+\cot(x_2)\cot(x_3)+\cot(x_3)\cot(x_1)=\cot(y_1)\cot(y_2)+\cot(y_2)\cot(y_3)+\cot(y_3)\cot(y_1)$$

K_T

$\uparrow$ trigonometric limit $(q \to 1)$

$$\left.\begin{array}{l} x_1x_2x_3=1 \\ y_1y_2y_3=1 \end{array}\right\} \Rightarrow \delta(x_1,y_2)\,\delta\!\left(x_2,\frac{1}{y_1}\right)+\delta(x_2,y_3)\,\delta\!\left(x_3,\frac{1}{y_2}\right)+\delta(x_3,y_1)\,\delta\!\left(x_1,\frac{1}{y_3}\right)=0$$

Ell_T

Thank you!