Algebraic Invariants of Hyperbolic 4-Orbifolds

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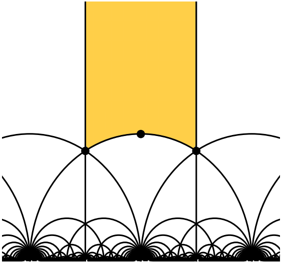
Modular Group



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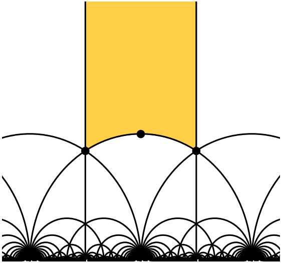

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- Visualize by taking fundamental domain of and gluing sides.

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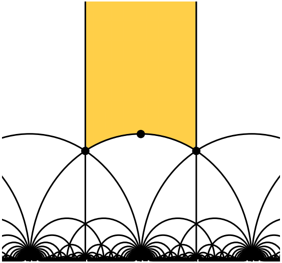

- Modular group $\quad = SL(2; Z)$ acts on the hyperbolic plane H^2.
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$
- We can construct an orbifold $H^2 = \;$.
- Visualize by taking fundamental domain of and gluing sides.
- Orbifold (rather than manifold) because i, $e^{\ i=3}$, $e^{2\ i=3}$ are fixed points.

Bianchi Groups

- Let K be an imaginary quadratic field, e.g. $\mathbb{Q}(\sqrt{-2})$.
- Let $\mathcal{O}_K$ be its ring of integers, e.g. $\mathbb{Z}[\sqrt{-2}]$.

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- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} . z = (az + b)(cz + d)^{-1}$, where $z = x + yi + zj$, a quaternion.

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- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} . p = (ap + b)(cp + d)^{-1}$, where $z = x + yi + zj$, a quaternion.
- We can again consider the orbifold $\mathbb{H}^3 / SL(2, \mathcal{O}_K)$.

Connecting Algebra and Geometry

$\mathfrak{o}_K$	$\mathbb{H}^3 / SL(2; \mathfrak{o}_K)$
Isomorphism class	Homotopy/isometry class

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Remark

General theory matches orders of split quaternion algebras and arithmetic 3-orbifolds.

Generalizing to Hyperbolic 4-Orbifolds:

Question

Can we get a similar theory for hyperbolic 4-orbifolds? That is, find some nice arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^4)$ *and relate algebraic invariants to geometric/topological ones?*

- Was there anything special about the 2 and 3-dimensional case?

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- Was there anything special about the 2 and 3-dimensional case?
- Yes: there are accidental isomorphisms $PSL(2;\mathbb{R}) = SO^+(2;1) = \mathrm{Isom}^0(\mathbb{H}^2)$ and $PSL(2;\mathbb{C}) = SO^+(3;1) = \mathrm{Isom}^0(\mathbb{H}^3)$.

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To proceed, we shall need a new accidental isomorphism with $SO^+(4;1) = \mathrm{Isom}^0(\mathbb{H}^4)$.

De nition

Given a ring R, an involution on R is a map $: R\ !\ R$ such that

1. $(x + y) = (x) + (y)$,
2. $(xy) = (y)\ (x)$,
3. $(\ (x)) = x$.

A homomorphism of rings with involution : $(R_1;\ _1)\ !\ (R_2;\ _2)$ is a ring homomorphism such that $_1 = \ _2$.

Examples of Involutions

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Mat(2; R) (R commutative),

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Mat(2; R) ((R;) ring with involution),

$$\wedge \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (d) & (b) \\ (c) & (a) \end{pmatrix}$$

Quaternion Algebras

Definition

A quaternion algebra over a field F is a central simple algebra H of dimension 4. If $\operatorname{char}(F) \neq 2$, this can always be expressed as the F-algebra generated by elements i, j satisfying $i^2 = a$, $j^2 = b$, $ij = -ji$ for some $a, b \in F^\times$. We write this as

$$H = \left(\frac{a, b}{F}\right) = \{x + yi + zj + tij \mid x, y, z, t \in F\}.$$

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Examples

$H_{\mathbb{R}} = \left(\frac{-1, -1}{\mathbb{R}}\right)$, the classical Hamilton quaternions.

$\mathrm{Mat}(2, F) = \left(\frac{1, 1}{F}\right)$.

Definition

Any involution on any central simple algebra A must restrict to an automorphism of the center F of either order 1 (*involution of first kind*) or order 2 (*involution of second kind*).

Involutions on Quaternion Algebras

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1. Standard (symplectic) involution: quaternion conjugation

$$\overline{x + yi + zj + tij} = x - yi - zj - tij$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

2. Orthogonal involutions: after change of variables

$$(x + yi + zj + tij)^{z} = x + yi + zj - tij\,:$$

Groups Constructed from Rings with Involution

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Let $(R; \;)$ be a ring with involution. We define the twisted special linear group

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One checks that

$$SL \ (2;R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat(2;R) \ \ a \ (b); \ c \ (d) \in R^+; \ a \ (d) \ \ b \ (c) = 1 \right\}$$

where R^+, R are the subsets of R on which acts as id and id, respectively.

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To Reiterate

$\mathbb{H}_{\mathbb{R}} = \{x + yi + zj + tij\}$, $i^2 = j^2 = -1$, $ij = -ji$.

$(x + yi + zj + tij)^{\ddagger} = x + yi + zj - tij$.

$$SL^{\ddagger}(2;\mathbb{H}_{\mathbb{R}}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2;\mathbb{H}_{\mathbb{R}}) \;\middle|\; ab^{\ddagger}, cd^{\ddagger} \in H^{+}; \; ad^{\ddagger} - bc^{\ddagger} = 1 \right\}$$

Isomorphism with Isometry Group

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We can now try to look for nice arithmetic subgroups.

Orders of Simple Algebras

Definition

Let $\mathfrak{o}$ be a Dedekind domain, F its field of fractions, and A a simple algebra over F. An *order* of A is a subring $\mathcal{O}$ which is also a $\mathfrak{o}$-lattice that is, a finitely-generated $\mathfrak{o}$-module such that $F\mathcal{O} = A$.

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Examples:

$$\mathfrak{o} = \mathbb{Z},\ F = \mathbb{Q},\ A = \mathbb{Q}(\sqrt{n}),\ \mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}\sqrt{n}.$$

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- $\mathfrak{o} = \mathbb{Z}$, $F = \mathbb{Q}$, $A = \left(\frac{-1, -1}{\mathbb{Q}}\right)$

$$\mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{1 + i + j + ij}{2}.$$

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Orders with Involution

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Let O be an order of a central simple algebra A with an involution . We say O is a -order if $(O) = O$. We say that O is a maximal -order if @ -order O^0) O.

Orders with Involution

Definition

Let $\mathcal{O}$ be an order of a central simple algebra A with an involution $\ddagger$. We say $\mathcal{O}$ is a $\ddagger$-order if $\ddagger(\mathcal{O}) = \mathcal{O}$. We say that $\mathcal{O}$ is a maximal $\ddagger$-order if $\nexists$ $\ddagger$-order $\mathcal{O}' \supsetneq \mathcal{O}$.

Why Do We Care?

- If $\mathcal{O}$ is a $\ddagger$-order, then $SL^{\ddagger}(2, \mathcal{O})$ is an arithmetic subgroup of the algebraic group $SL^{\ddagger}(2, A)$.
- If H is a rational, definite quaternion algebra, then $SL^{\ddagger}(2, \mathcal{O})$ is an arithmetic subgroup of $\mathrm{Isom}^0(\mathbb{H}^4)$.

Examples of Orders with Involution

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$$O = Z \quad Zi \quad Zj \quad Z\frac{1 + i + j + ij}{2}:$$

$$H = \frac{2; \quad 3}{Q}, \ (x + yi + zj + tij)^z = x + yi + zj \quad tij.$$

$$O = Z \quad Zi \quad Z\frac{1 + j}{2} \quad Z\frac{i + ij}{2}$$

$$Z \quad Zi \quad Z\frac{1 + j}{2} \quad Z\frac{3 + 3i + j + ij}{6}:$$

Determining Maximality

Theorem (S. 2017)

Let K be a local or global field. Let H be a quaternion algebra over K with orthogonal involution z. Then $\mathcal{O}$ is a maximal z-order if and only if it is a z-order with discriminant $\operatorname{disc}(\mathcal{O}) = \operatorname{disc}(H) \setminus \quad (\operatorname{disc}(z))$.

Remark

For local fields, better results are available for classifying isomorphism classes.

Connecting Algebra and Geometry, Part II

O	$H^4 = SL^z(2; O)$	
Isomorphism class of $Mat(2; O); \hat{z}$	Homotopy/isometry class	18
Maximality (among z-orders)	Minimal volume (among arithmetic orbifolds)	19
Class number	Number of cusps	20,22
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Open Question

Do all arithmetic 4-orbifolds correspond to -orders inside $Mat(2; H_R)$?

Remarks about Isomorphism Classes

For 3-orbifolds, the isomorphism class of o_K mattered. This is because $(\mathrm{Mat}(2; o_K); y) = (\mathrm{Mat}(2; o_{K^0}); y)$ if and only if $o_K = o_{K^0}$.

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- $\mathrm{Mat}(2; O) = \mathrm{Mat}(2; O^0)$ 6) $O = O^0$. (Examples due to Chatters 1996)
- $(\mathrm{Mat}(2; O); \hat{z}) = (\mathrm{Mat}(2; O^0); \hat{z})$ 6) $O = O^0$. (S. 2020)
- Conversely, $O = O^0$ 6) $(\mathrm{Mat}(2; O); \hat{z}) = (\mathrm{Mat}(2; O^0); \hat{z})$. (S. 2020)

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Conversely, $O = O' \not\Rightarrow (\mathrm{Mat}(2; O); \hat{z}) = (\mathrm{Mat}(2; O'); \hat{z})$. (S. 2020)

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$(\mathrm{Mat}(2; \mathcal{O}); \ddagger) \cong (\mathrm{Mat}(2; \mathcal{O}'); \ddagger) \not\Rightarrow \mathcal{O} \cong \mathcal{O}'$. (S. 2020)

Conversely, $\mathcal{O} \cong \mathcal{O}' \not\Rightarrow (\mathrm{Mat}(2; \mathcal{O}); \ddagger) \cong (\mathrm{Mat}(2; \mathcal{O}'); \ddagger)$. (S. 2020)

Rough outline of proof:

- Prove $\mathbb{Z}[SL^{\ddagger}(2; \mathcal{O})] = \mathrm{Mat}(2; \mathcal{O})$ using strong approximation theorem for algebraic groups.

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- Note that $SL^z(2; O_1)$ conjugate to $SL^z(2; O_2)$ if and only if $H^4 = SL^z(2; O_1)$ is isometric to $H^4 = SL^z(2; O_2)$ by the Mostow rigidity theorem.

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Remarks About Maximality

Bianchi groups $SL(2; o_K)$ are maximal among arithmetic subgroups of $SL(2; K)$. Extended Bianchi groups needed to get maximal discrete subgroups of $SL(2; \mathbb{C})$.

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  - If $SL^z(2; O)$, then it is an $\hat{z}$-order of $Mat(2; H)$ containing $Z[SL^z(2; O)] = Mat(2; O)$. Check that $Mat(2; O)$ is $\hat{z}$-maximal.

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Remarks about Ideals

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Theorem (S. 2019)

Let H be a quaternion algebra over an algebraic number field K with orthogonal involution $\ddagger$. Let $\mathcal{O}$ be a $\ddagger$-order of H and let I be an invertible right ideal of $\mathcal{O}$. Then $I = x\mathcal{O} + y\mathcal{O}$ for some $x, y \in \mathcal{O}$ such that $xy^{\ddagger} \in \mathcal{O}^{+}$.

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Corollary

For invertible left ideals, we can write $I = \mathcal{O}x + \mathcal{O}y$ for some $x, y \in \mathcal{O}$ such that $xy^{-1} \in H^+$.

Remarks about Ideals, Part II

Theorem (S. 2019)

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Rough Outline of Proof

1. Show that for almost all prime ideals $\mathfrak{p}$, when we pass to the localization $\mathcal{O}_{\mathfrak{p}}$, $I_{\mathfrak{p}} = x_{\mathfrak{p}}\mathcal{O}$ for some $x_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^{+}$. This uses the fact that H^{+} is three-dimensional and Chevalley-Warning.
2. Show there exists $z \in \mathcal{O}$ such that $J = zI$ is generated by one element in $t \in \mathfrak{o}_K$ for every remaining bad place. Replace I by J WLOG.
3. Use strong approximation theorem on $SL(2, H)$ to prove that $J \mod (t) = (x_{\mathfrak{p}})_{\mathfrak{p}} \mod (t) = (x') \mod (t)$ for some $x' \in \mathcal{O}^{+}$. Thus, $J = x'\mathcal{O} + t\mathcal{O}$.

Remarks about Class Numbers and Cusps

The (left) class set of $\mathcal{O}$ is $\mathrm{Cls}_L(\mathcal{O})$: the equivalence classes of (invertible) left $\mathcal{O}$-ideals modulo right multiplication by $\mathcal{O}^\times$. (This is not a group anymore.)

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Theorem (S. unpublished)

Let H be a definite, rational quaternion algebra with an orthogonal involution z. Let $\mathcal{O}$ be a maximal z-order of H. There is a well-defined bijection

$$\Phi : \quad H^+ \cup \{\infty\} / \mathrm{SL}^z(2;\mathcal{O}) \to \mathrm{Cls}_L(\mathcal{O})$$
$$xy^{-1} \mapsto [\mathcal{O}x + \mathcal{O}y].$$

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Corollary

The number of cusps of $\mathbb{H}^4/\mathrm{SL}^\ddagger(2;\mathcal{O})$ is the class number of $\mathcal{O}$.

Remarks about Class Numbers and Cusps, Part II

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Rough Outline of Proof

1. Surjectivity follows from remarks about generators of left ideals.
2. $SL^{\ddagger}(2;\mathbb{H})$ acts transitively on the bers of this map. To prove injectivity, su ces to consider $\ ^{1}([\mathcal{O}])$.
3. Given $x, y \in \mathcal{O}$ such that $\mathcal{O}x + \mathcal{O}y = \mathcal{O}$ and $xy^{\ 1} \in \mathbb{H}^+$, explicitly construct $\ \in SL^{\ddagger}(2;\mathcal{O})$ such that $:1 \ = xy^{\ 1}$.

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Remarks about Volume

Many proofs of volume of $H^3 =$ where $= SL(2; o_K)$.

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Many proofs of volume of H^3 / Γ where $\Gamma = SL(2; o_K)$.

One possible approach: define Eisenstein series

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and compute Fourier coefficients.

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- If we take $=$ $SL^z(2; O)$, then we can de ne Eisenstein series

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- Maybe a similar proof is possible?

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Remarks about z-Euclidean Algorithm

Definition

Let R be a ring without zero divisors. Suppose that there exists a well-ordered set W and a function $\phi : R \to W$ such that for all $a, b \in R$ such that $b \neq 0$, there exists $q \in R$ such that $\phi(a - bq) < \phi(b)$. Then we say that R is a *Euclidean ring* with stathm ϕ.

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Definition

Let $(R, \ddagger)$ be a ring with involution, without zero divisors. Suppose that there exists a well-ordered set W and a function $\phi : R \to W$ such that for all $a, b \in R$ such that $b \neq 0$ and $a(b)^{\ddagger} \in R^+$, there exists $q \in R^+$ such that $\phi(a - bq) < \phi(b)$. Then we say that R is a **‡-Euclidean ring** with stathm ϕ.

Remarks about -Euclidean Algorithm, Part II

De nition

If $(R; \;)$ is -Euclidean, we can de ne a function

$$f \;\; : R \;\; R \to \;\; R$$
$$(a; b) \mapsto (q; a \;\; bq)$$

where $(\; a \;\; bq) < \; (\; b)$.

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Algorithm (S. 2019)

On an input of $a; b \in R$ such that $a \ (b) \in R^+$, returns sequences $r_i; s_i; t_i \quad R$ with $0 \quad i \quad k + 1$ and:

1. $r_{k+1} = 0$.
2. r_k is a right GCD of a and b.
3. $a \ (s_i) \quad b \ (t_i) = r_i$ for all i, and $s_i \ (t_i) \in R^+$.

Remarks about z-Euclidean Algorithm, Part III

```
procedure (a,b)
    r_list ← [a; b]
    s_list ← [1; 0]
    t_list ← [0; 1]
    while r_list[−1] ≠ 0 do
        r_{i−2} ← r_list[−2]
        r_{i−1} ← r_list[−1]
        (q; r_i) ← f(r_{i−2}; r_{i−1})
        append(r_list; r_i)
        s_{i−2} ← s_list[−2]
        s_{i−1} ← s_list[−1]
        s_i ← s_{i−2} − q s_{i−1}
        append(s_list; s_i)
        t_{i−2} ← t_list[−2]
        t_{i−1} ← t_list[−1]
        t_i ← t_{i−2} − q t_{i−1}
        append(s_list; t_i)
    return r_list; s_list; t_list
```

Remarks about z-Euclidean Algorithm, Part IV

Corollary

If $(\mathcal{O}, z)$ is a z-Euclidean ring, then it has class number 1.

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If $(\mathcal{O}; z)$ is a z-Euclidean ring, then it has class number 1.

Brief Proof

For any left ideal I, write it as $I = x\mathcal{O} + y\mathcal{O}$ with $xy^z \in \mathcal{O}^+$. Run the algorithm to find the GCD of x and y.

Remarks about z-Euclidean Algorithm, Part IV

Corollary

If $(O; z)$ is a z-Euclidean ring, then it has class number 1.

Brief Proof

For any left ideal I, write it as $I = xO + yO$ with $xy^z\ 2\ O^+$. Run the algorithm to find the GCD of x and y.

Corollary

If $(O; z)$ is a z-Euclidean ring, then $SL^z(2; O)$ is generated by elementary matrices

$$\begin{array}{cc} 1 & \\ 0 & 1 \end{array} ; \quad \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} ; \quad \begin{array}{cc} u & 0 \\ 0 & (u^z)^{\ 1} \end{array} :$$

Remarks about z-Euclidean Algorithm, Part V

How to Construct Dirichlet Domain? (Very Brief Summary)

1. Choose point $z = \quad + tij \; 2 \; \mathbb{H}^4$ with $\quad 2 \; H^+$ such that Stab $_1(\quad) = f \quad Ig$ and $t > 0$ sufficiently large.

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5. Compute the diameter of this set minus the cusp. Use the z-Euclidean algorithm to search through all 2 $SL^z(2; O)$ such that $:z$ is within this diameter from z—there will only be finitely many.

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6. Cut away geodesic planes corresponding to such . You have a bona fide Dirichlet domain for .

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Q/A

Thank you very much for inviting me to give a talk!