Algebraic Invariants of Hyperbolic 4-Orbifolds

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- We can construct an orbifold $\mathbb{H}^2/\Gamma.$
- Visualize by taking fundamental domain of Γ and gluing sides.
- Orbifold (rather than manifold) because *i*, $e^{\pi i/3}$, $e^{2\pi i/3}$ are fixed points.

Bianchi Groups



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- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $p = (ap+b)(cp+d)^{-1}$, where z = x + yi + zj, a quaternion.
- We can again consider the orbifold ℍ³/SL(2, 𝒪_K).

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Remark

General theory matches orders of split quaternion algebras and arithmetic 3-orbifolds.

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Hyperbolic 4-Orbifolds

Question

Can we get a similar theory for hyperbolic 4-orbifolds? That is, find some nice arithmetic subgroups of $Isom(\mathbb{H}^4)$ and relate algebraic invariants to geometric/topological ones?

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- Was there anything special about the 2 and 3-dimensional case?
- Yes: there are accidental isomorphisms $PSL(2, \mathbb{R}) \cong SO^+(2, 1) \cong \text{Isom}^0(\mathbb{H}^2)$ and $PSL(2, \mathbb{C}) \cong SO^+(3, 1) \cong \text{Isom}^0(\mathbb{H}^3)$.

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- To proceed, we shall need a new accidental isomorphism with $SO^+(4,1) \cong Isom^0(\mathbb{H}^4).$

Given a ring R, an *involution* on R is a map $\sigma : R \rightarrow R$ such that

$$(x + y) = \sigma(x) + \sigma(y),$$

A homomorphism of rings with involution $\phi : (R_1, \sigma_1) \to (R_2, \sigma_2)$ is a ring homomorphism such that $\phi \circ \sigma_1 = \sigma_2 \circ \phi$.

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• Mat(2, R) ((R, σ) ring with involution),

$$\hat{\sigma} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sigma(d) & -\sigma(b) \\ -\sigma(c) & \sigma(a) \end{pmatrix}$$

A quaternion algebra over a field F is a central simple algebra H of dimension 4. If char $(F) \neq 2$, this can always be expressed as the F-algebra generated by elements i, j satisfying $i^2 = a, j^2 = b, ij = -ji$ for some $a, b \in F^{\times}$. We write this as

$$H = \left(\frac{a,b}{F}\right) = \left\{x + yi + zj + tij|x, y, z, t \in F\right\}.$$

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Examples

Any involution σ on any central simple algebra A must restrict to an automorphism of the center F of either order 1 (involution of *first kind*) or order 2 (involution of *second kind*).

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Standard (symplectic) involution: quaternion conjugation

$$\overline{x + yi + zj + tij} = x - yi - zj - tij$$
$$\begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \end{pmatrix}$$

② Orthogonal involutions: after change of variables

$$(x + yi + zj + tij)^{\ddagger} = x + yi + zj - tij.$$

Let (R,σ) be a ring with involution. We define the twisted special linear group

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One checks that

$$egin{aligned} SL^{\sigma}(2,R) &= iggl\{ inom{a}{c} inom{b}{c} inom{d}{d} iggr\in \mathsf{Mat}(2,R) igg| a\sigma(b), \ c\sigma(d) \in R^+, \ a\sigma(d) - b\sigma(c) = 1 iggr\} \end{aligned}$$

where R^+ , R^- are the subsets of R on which σ acts as *id* and *-id*, respectively.

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To Reiterate

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$$H_{\mathbb{R}} = \{x + yi + zj + tij\}, i^2 = j^2 = -1, ij = -ji.$$

•
$$(x + yi + zj + tij)^{\ddagger} = x + yi + zj - tij.$$

$$egin{aligned} SL^{\ddagger}(2,\mathcal{H}_{\mathbb{R}}) &= iggl\{iggl(egin{aligned} a & b\ c & d \end{pmatrix} \in \operatorname{Mat}(2,\mathcal{H}_{\mathbb{R}}) iggr| ab^{\ddagger}, cd^{\ddagger} \in \mathcal{H}^+, \ ad^{\ddagger} - bc^{\ddagger} = 1iggr\} \end{aligned}$$

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- We can now try to look for nice arithmetic subgroups.

Let σ be a Dedekind domain, F its field of fractions, and A a simple algebra over F. An *order* of A is a subring \mathcal{O} which is also a σ -lattice—that is, a finitely-generated σ -module such that $F\mathcal{O} = A$.
Let o be a Dedekind domain, F its field of fractions, and A a simple algebra over F. An *order* of A is a subring O which is also a o-lattice—that is, a finitely-generated o-module such that FO = A.

Examples:

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$$\mathfrak{o} = \mathbb{Z}$$
, $F = \mathbb{Q}$, $A = \mathbb{Q}(\sqrt{-n})$, $\mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}\sqrt{-n}$.

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• $\mathfrak{o} = \mathbb{Z}, F = \mathbb{Q}, A = \left(\frac{-1,-1}{\mathbb{Q}}\right)$
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• $\mathfrak{o} = \mathbb{Z}, F = \mathbb{Q}, A = \left(\frac{-2, -3}{\mathbb{Q}}\right)$
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Let \mathcal{O} be an order of a central simple algebra A with an involution σ . We say \mathcal{O} is a σ -order if $\sigma(\mathcal{O}) = \mathcal{O}$. We say that \mathcal{O} is a maximal σ -order if \nexists σ -order $\mathcal{O}' \supseteq \mathcal{O}$.

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Why Do We Care?

- If *O* is a *σ*-order, then SL^σ(2, *O*) is an arithmetic subgroup of the algebraic group SL^σ(2, A).
- If H is a rational, definite quaternion algebra, then SL[‡](2, O) is an arithmetic subgroup of Isom⁰(𝔄⁴).

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 $\subset \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+j}{2} \oplus \mathbb{Z}\frac{3+3i+j+ij}{6}.$

Theorem (S. 2017)

Let K be a local or global field. Let H be a quaternion algebra over K with orthogonal involution \ddagger . Then \mathcal{O} is a maximal \ddagger -order if and only if it is a \ddagger -order with discriminant disc(\mathcal{O}) = disc(H) $\cap \iota$ (disc(\ddagger)).

Remark

For local fields, better results are available for classifying isomorphism classes.

Connecting Algebra and Geometry, Part II

\mathcal{O}	$\mathbb{H}^4/SL^{\ddagger}(2,\mathcal{O})$	
Isomorphism class of $\left(Mat(2, \mathcal{O}), \hat{\ddagger}\right)$	Homotopy/isometry class	18
Maximality (among ‡-orders)	Minimal volume (among arithmetic orbifolds)	19
Class number	Number of cusps	20,22
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Open Question

Do all arithmetic 4-orbifolds correspond to σ -orders inside Mat(2, $H_{\mathbb{R}}$)?

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 - Prove that $\left(\mathbb{Z}[SL^{\ddagger}(2, \mathcal{O}_1)], \ddagger\right) \cong \left(\mathbb{Z}[SL^{\ddagger}(2, \mathcal{O}_2)], \ddagger\right)$ if and only if $SL^{\ddagger}(2, \mathcal{O}_1)$ conjugate to $SL^{\ddagger}(2, \mathcal{O}_2)$. This uses the Skolem-Noether theorem.

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 - Solution Note that SL[‡](2, O₁) conjugate to SL[‡](2, O₂) if and only if ℍ⁴/SL[‡](2, O₁) is isometric to ℍ⁴/SL[‡](2, O₂) by the Mostow rigidity theorem.

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 - If Γ ⊃ SL[‡](2, O), then it is an [‡]-order of Mat(2, H) containing Z[SL[‡](2, O)] = Mat(2, O). Check that Mat(2, O) is [‡]-maximal.

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Theorem (S. 2019)

Let H be a quaternion algebra over an algebraic number field K with orthogonal involution \ddagger . Let \mathcal{O} be a \ddagger -order of H and let I be an invertible right ideal of \mathcal{O} . Then $I = x\mathcal{O} + y\mathcal{O}$ for some $x, y \in \mathcal{O}$ such that $xy^{\ddagger} \in \mathcal{O}^+$.

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Corollary

For invertible left ideals, we can write I = Ox + Oy for some $x, y \in O$ such that $xy^{-1} \in H^+$.

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Rough Outline of Proof

- Show that for almost all prime ideals p, when we pass to the localization O_p, I_p = x_pO for some x_p ∈ O⁺_p. This uses the fact that H⁺ is three-dimensional and Chevalley-Warning.
- ② Show there exists $z \in O$ such that J = zI is generated by one element in $t \in \mathfrak{o}_K$ for every remaining bad place. Replace *I* by *J* WLOG.
- Use strong approximation theorem on SL(2, H) to prove that J mod (t) = (x_p)_p mod (t) = (x') mod (t) for some x' ∈ O⁺. Thus, J = x'O + tO.

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Theorem (S. unpublished)

Let H be a definite, rational quaternion algebra with an orthogonal involution \ddagger . Let \mathcal{O} be a maximal \ddagger -order of H. There is a well-defined bijection

$$\Psi: \left(H^{+} \cup \{\infty\}\right) / SL^{\ddagger}(2, \mathcal{O}) \to Cls_{L}(\mathcal{O})$$
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Corollary

The number of cusps of $\mathbb{H}^4/SL^{\ddagger}(2, \mathcal{O})$ is the class number of \mathcal{O} .
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Rough Outline of Proof

- **1** Surjectivity follows from remarks about generators of left ideals.
- SL[‡](2, H) acts transitively on the fibers of this map. To prove injectivity, suffices to consider Ψ⁻¹([O]).
- Given x, y ∈ O such that Ox + Oy = O and xy⁻¹ ∈ H⁺, explicitly construct γ ∈ SL[‡](2, O) such that γ.∞ = xy⁻¹.

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• Maybe a similar proof is possible? Return to index. 17

Definition

Let *R* be a ring without zero divisors. Suppose that there exists a well-ordered set *W* and a function $\Phi : R \to W$ such that for all $a, b \in R$ such that $b \neq 0$, there exists $q \in R$ such that $\Phi(a - bq) < \Phi(b)$. Then we say that *R* is a *Euclidean ring*, with *stathm* Φ .

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Let (R, σ) be a ring with involution, without zero divisors. Suppose that there exists a well-ordered set W and a function $\Phi : R \to W$ such that for all $a, b \in R$ such that $b \neq 0$ and $a\sigma(b) \in R^+$, there exists $q \in R^+$ such that $\Phi(a - bq) < \Phi(b)$. Then we say that R is a σ -Euclidean ring, with stathm Φ .

Definition

If (R, σ) is σ -Euclidean, we can define a function

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Algorithm (S. 2019)

On an input of $a, b \in R$ such that $a\sigma(b) \in R^+$, returns sequences $r_i, s_i, t_i \subset R$ with $0 \le i \le k + 1$ and:

1
$$r_{k+1} = 0.$$

- 2 r_k is a right GCD of a and b.
- 3 $a\sigma(s_i) b\sigma(t_i) = r_i$ for all i, and $s_i\sigma(t_i) \in R^+$.

1:	procedure (a,b)	10:	$s_{i-2} \leftarrow s_{list}[-2]$
2:	$r_{list} \leftarrow [a, b]$	11:	$s_{i-1} \leftarrow s_{list}[-1]$
3:	$s_{list} \leftarrow [1,0]$	12:	$s_i \leftarrow s_{i-2} - qs_{i-1}$
4:	$t_{list} \leftarrow [0, -1]$	13:	$append(s_{list}, s_i)$
5:	while $r_{list}[-1] \neq 0$ do	14:	$t_{i-2} \leftarrow t_{list}[-2]$
6:	$r_{i-2} \leftarrow r_{list}[-2]$	15:	$t_{i-1} \leftarrow t_{list}[-1]$
7:	$r_{i-1} \leftarrow r_{list}[-1]$	16:	$t_i \leftarrow t_{i-2} - qt_{i-1}$
8:	$(q, r_i) \leftarrow f_{\Phi}(r_{i-2}, r_{i-1})$	17:	$append(s_{list}, t_i)$
9:	$append(r_{list}, r_i)$	18:	return $r_{\text{list}}, s_{\text{list}}, t_{\text{list}}$

Corollary

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Corollary

If (O,‡) is a ‡-Euclidean ring, then SL^‡(2,O) is generated by elementary matrices

$$\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & (u^{\ddagger})^{-1} \end{pmatrix}.$$

How to Construct Dirichlet Domain? (Very Brief Summary)

• Choose point $z = \alpha + tij \in \mathbb{H}^4$ with $\alpha \in H^+$ such that $\operatorname{Stab}_{\Gamma_{\infty}}(\alpha) = \{\pm I\}$ and t > 0 sufficiently large.

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- Cut away geodesic planes corresponding to such $\gamma.$ You have a bona fide Dirichlet domain for $\Gamma.$

Thank you very much for inviting me to give a talk!