Towards an Exceptional Knot Polynomial

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jt with S. Morrison & D. Thurston

Diagrams for Rep(G)

Strands labelled by representations

Vertices labelled by morphisms

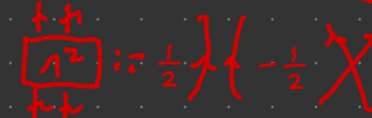
$$
\begin{array}{c}
V \\
\uparrow \mathrm{id} \otimes \mathrm{ev} \\
V \otimes V \otimes V^* \\
\uparrow \mathrm{swap} \otimes \mathrm{id} \\
V \otimes V \otimes V^* \\
\uparrow \mathrm{id} \otimes \varphi \otimes \mathrm{id} \\
V \otimes V \otimes V \otimes V^*
\end{array}
$$

Rep(GLn) and Schur-Weyl Duality

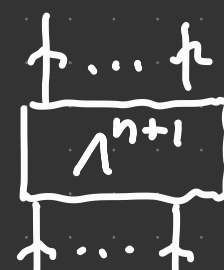
Strand is standard rep.

Tangles span all maps 

Only relations:

Here 

 $= n$



$=$ zero

Deligne's GL_t Family

GL_t is the completion of the symmetric $\otimes$-category presented by

One generating obj.
No generating morphisms
Relation:  $= t$

O_t is the same but with unoriented strands

More about GL_t

- For $t \notin \mathbb{Z}$ it's semisimple and looks like $\mathrm{Rep}(GL_N)$ for $N \gg 0$.
- For $n \in \mathbb{N}$ it has $\mathrm{Rep}(GL_n)$ as a quotient
- Also has $\mathrm{Rep}(GL(n+m|m))$ as a quotient
- Can make $GL_n^{ab} = \varprojlim \mathrm{Rep}(GL(n+m|m))$

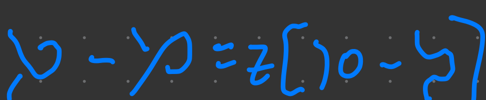
  (see Entova-Aizenbud, Hinich, Serganova)

Quantum O_t and the Kauffman Polynomial

Want a braided $\otimes$-category "like" O_t

Relation between 

wlog relation is an eigenvector for rotation



$a - a^{-1} = z(d-1)$

$$\boxed{d = 1 + \frac{a - a^{-1}}{z}}$$

Following Kauffman, Kuperberg, Kazhdan-Wenzl

Deligne's Exceptional Series

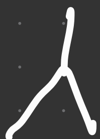
Adjoint reps of

$PSL(2)$, $PSL(3) \rtimes \mathbb{Z}/2\mathbb{Z}$, G_2, $PSO(8) \rtimes S_3$,

F_4, $E_6 \rtimes \mathbb{Z}/2\mathbb{Z}$, E_7, E_8

Have patterns that look like they come from a continuous 1-parameter family of symmetric $\otimes$-cats.

Relations for Exceptional Family

 is the bracket 

Convention

Jacobi: 

Exceptional: 

Cvitanovic

Vogel

Forced by $\dim \mathrm{Hom}(\mathrm{Sym}^4 \mathfrak{g}, 1) = 1$.

Conjectures

<u>Sufficiency</u>: For d generic any closed trivalent graph simplifies to a multiple of empty.

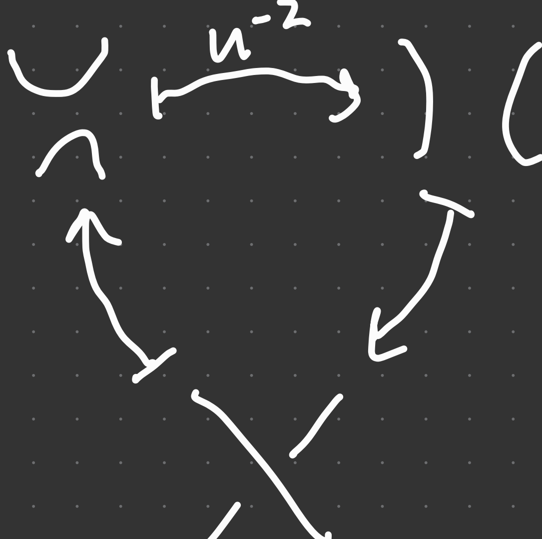
<u>Consistency</u>: For d generic any two ways of simplifying the same diagram give the same answer.

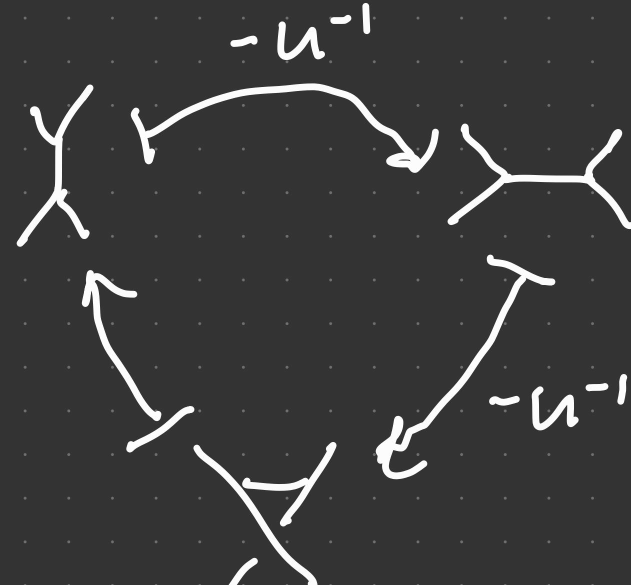
Together give an Exceptional Family!

Quantum Version

Need some relation between

Tetrahedral Symmetry

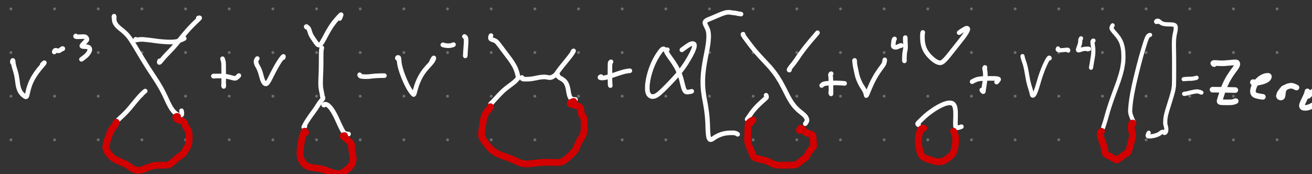




QExc version of Jacobi

- R^3 acts by scalar u^{-2}
- R acts diagonalizably on span
- WLOG relation is a λ-eigenvector $\lambda^3=u^{-2}$
- Change variables to v with $v^6=u$ and $\lambda=v^{-4}$

Easy Consequences



$$v^{-3}v^{12}b - v^{-1}b + \alpha v^{12} + \alpha v^{4} d + \alpha v^{-4} = 0$$

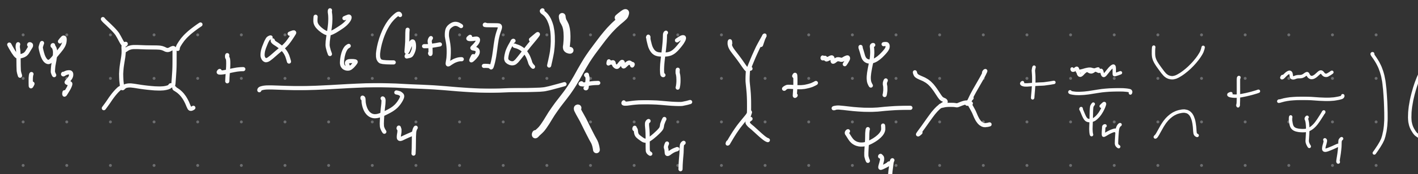
$$\alpha d = -v^{5}b + v^{-5}b - \alpha v^{8} - \alpha v^{-8}$$

$$\boxed{d = \frac{v^{5} - v^{-5}}{b} - \alpha\left(v^{8} + v^{-8}\right)}$$

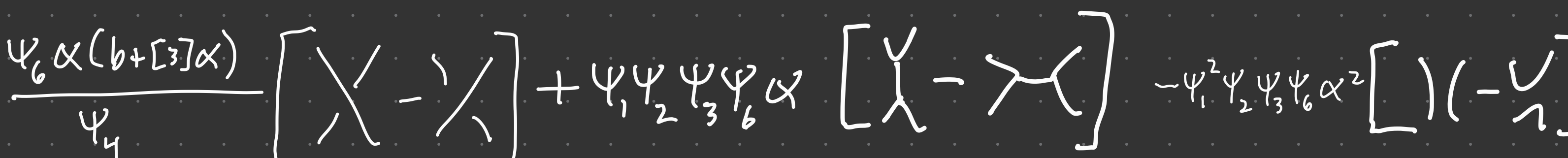
Similar formula 

QExc Square and Crossing

Attach  to QEJacobi and R-symmetrize



Rotationally symmertrize



= Zero

Main Theorem

If $\mathcal{C}$ is a ribbon category with an obj X and a map $X \otimes X \to X$ s.t. $\dim \mathrm{In}(X^{\otimes k})$ starts $1, 0, 1, 1, 5$ then it satisfies QE Jacobi for some v. If v is not a 10^{th} or 12^{th} root of unity it also satisfies QESquare, QECrossing

Conjectures

<u>Sufficiency</u>: These relations allow one to evaluate any braided trivalent graph

<u>Consistency</u>: Any two evaluations agree

Together give 2-parameter knot polynomial

Other Topics

- Classical $\Rightarrow$ Quantum via Kontsevich $\int$
- Roots of unity: $1^{st} \rightsquigarrow$ Exceptional
  $3^{rd}/6^{th} \rightsquigarrow G_2$ family
  $4^{th} \rightsquigarrow F_4$ family
  $12^{th} \rightsquigarrow S_+$ and ???
  $5^{th} \rightsquigarrow$ ABA variant?
- Works for actual $\mathfrak{g}$ Etingof-Neshveyev

Calculations

- Affine braid group action on $\mathrm{Inv}(X^{\otimes 6})$
- Knot invariant calculated when Conway girth ≤ 6. Includes all ≤ 12 crossing knots

  Torus knots in Mironov Mkrtchyan Morozov
- Quantum dimensions of simples in $X^{\otimes 3}$

Classically by Cohen-de Man, $X^{\otimes 2}$ by Tuba-Wenzl