Theta Surfaces

Bernd Sturmfels
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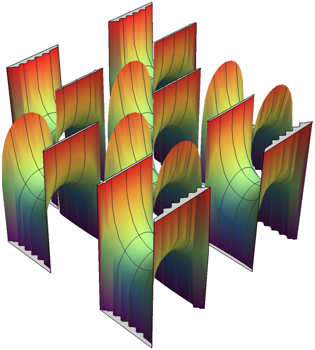

Joint Work with Daniele Agostini,
Türkü Özlüm Çelik and Julia Struwe

Scherk's Minimal Surface

Our running example is the surface in $\mathbb{R}^3$ given by

$$\sin(X) \; - \; \sin(Y) \cdot \exp(Z) \; = \; 0.$$

This theta surface has the parametric representation

$$\begin{aligned} (X, Y, Z) \;\; = \;\; & \qquad \big(\arctan(s), 0\,, \; \log(s) - \log(s^2{+}1)/2\,\big) \\ & + \;\; \big(\,0\,, \arctan(t)\,, -\log(t) + \log(t^2{+}1)/2\,\big). \end{aligned}$$

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A second parametrization:

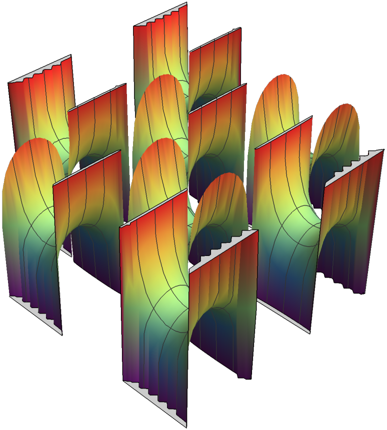
$$\begin{aligned} X &= \arctan(u) \; + \; \arctan(v) && = \; \arctan\left(\tfrac{u+v}{1-uv}\right) \\ Y &= \arctan(5u) \; + \; \arctan(5v) && = \; \arctan\left(\tfrac{5u+5v}{1-25uv}\right) \\ Z &= \tfrac{1}{2}\log\left(\tfrac{1+(5u)^2}{5(1+u^2)}\right) \; + \; \tfrac{1}{2}\log\left(\tfrac{1+(5v)^2}{5(1+v^2)}\right) \end{aligned}$$

This theta surface is derived from the following quartic curve in the *complex projective plane* $\mathbb{P}^2$:

$$xy(x^2 + y^2 + z^2) \; = \; 0.$$

Scherk's Minimal Surface

$$\sin(X) = \sin(Y) \cdot \exp(Z)$$

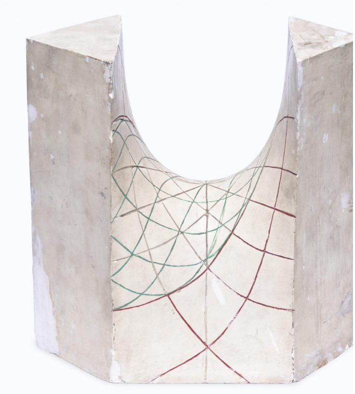


$$(X, Y, Z) = \left(\arctan(s), 0, \log(s) - \log(s^2+1)/2\right) + \left(0, \arctan(t), -\log(t) + \log(t^2+1)/2\right).$$

Double Translation Surfaces

An analytic surface $\mathcal{S}$ in $\mathbb{R}^3$ or $\mathbb{C}^3$ is a *double translation surface* if it has two distinct representations as Minkowski sum of curves:

$$\mathcal{S} \quad = \quad \mathcal{C}_1 + \mathcal{C}_2 \; = \; \mathcal{C}_3 + \mathcal{C}_4$$



Sophus Lie: *Bestimmung aller Flächen die in mehrfacher Weise durch Translationsbewegung einer Curve erzeugt werden*, Archiv Math. (1882).

Leipzig

Felix Klein held the professorship for geometry at Leipzig until 1886 when he moved to Göttingen. Sophus Lie was appointed to be Klein's successor and he moved from Christiania (Oslo) to Leipzig.

In his first years, Lie was busy with completing his major work *Theory of Transformation Groups*. Thereafter, double translation surfaces moved back in the focus of his teaching and research.

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In 1892 Lie published an article explaining how these surfaces can be parametrized by abelian integrals. In 1895 Henri Poincaré gave his proof of the relationship to Abel's Theorem. This led to the idea that these **surfaces** are **theta** divisors of 3-dim'l Jacobians.

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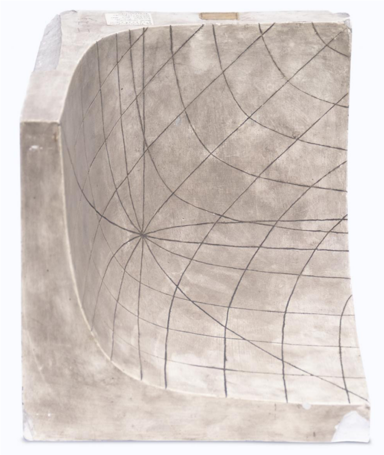
Lie supervised two doctoral dissertations on this topic at Leipzig. His students constructed a series of **plaster models** that visualize the diverse shapes exhibited by theta surfaces. It is surprising that the models were commissioned by Lie, who, unlike Klein, had not been known so far for an engagement in popularizing geometry.

Students of Sophus Lie



Richard Kummer: *Die Flächen mit unendlichvielen Erzeugungen durch Translation von Kurven*, Dissertation, Universität Leipzig, 1894.

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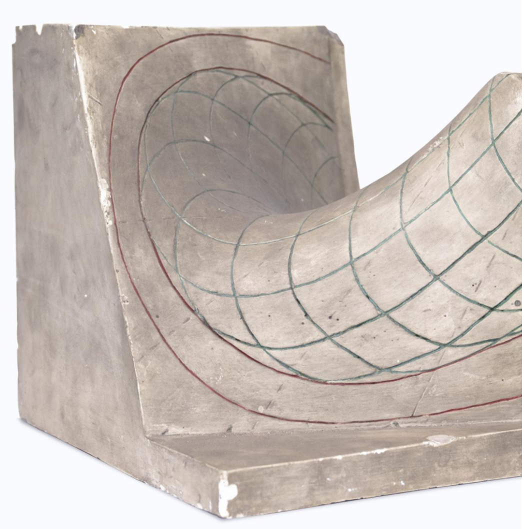
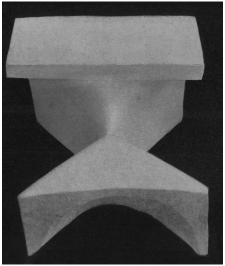


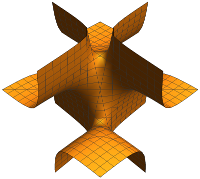
Georg Wiegner: *Über eine besondere Klasse von Translationsflächen*, Dissertation, Universität Leipzig, 1893.

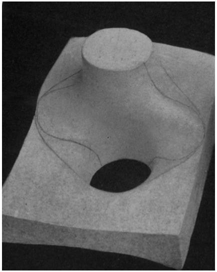
West Virginia

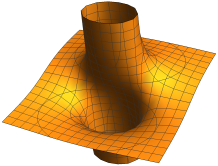
Johan "John" Arndt Eiesland (27 January 1867, Ny-Hellesund, Norway – 11 March 1950, Morgantown, West Virginia) was a Norwegian-American mathematician, specializing in differential geometry.

Eiesland immigrated to the USA in 1888 after completing his secondary education in Christiansand. He received his bachelor's degree from the University of South Dakota in 1891. He was a mathematics professor from 1895 to 1903 at Thiel College in Pennsylvania. On a leave of absence, he studied mathematics as a Johns Hopkins Scholar (1897–1898) at Johns Hopkins University, where he received his Ph.D. in 1898.[1] From 1903 to 1907 he was a mathematics instructor at the United States Naval Academy. In 1907 he became a professor of mathematics at West Virginia University and the chair of the mathematics department from 1907 to 1938,









John Eiesland (1908-09) completed the classification initiated by Kummer and Wiegner. He also constructed plaster models. These were donated to the collection at John Hopkins University.

Abelian Integrals

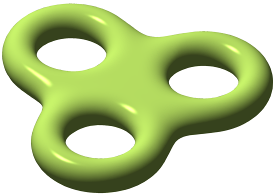
Consider a quartic curve $\mathcal{Q}$ in the projective plane $\mathbb{P}^2_{\mathbb{C}}$, defined by

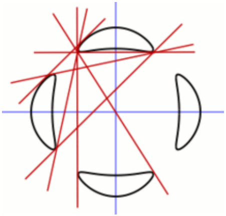
$$q(x, y) \quad = \quad \sum_{i+j\leq 4} c_{ij}\, x^i y^j .$$

The space of holomorphic differentials $H^0(\mathcal{Q}, \Omega^1_{\mathcal{Q}})$ has the basis

$$\omega_1 \;=\; \frac{x}{q_y} dx\,, \quad \omega_2 \;=\; \frac{y}{q_y} dx\,, \quad \omega_3 \;=\; \frac{1}{q_y} dx.$$

Note: $\mathcal{Q}$ is the canonical embedding of the underlying Riemann surface.





We obtain a complex number $\int_p^r \omega_i$ if we integrate along a path on $\mathcal{Q}$ with end points p and r. If the path is real then $\int_p^r \omega_i$ is real.

Two Norwegians

Fix a line $\mathcal{L}(0)$ in $\mathbb{P}^2$ that intersects $\mathcal{Q}$ in four points $p_1(0), p_2(0)$, $p_3(0), p_4(0)$. For $p_j \in \mathcal{Q}$ close to $p_j(0)$, we obtain the space curve

$$\Omega_j(p_j) \;=\; (\Omega_{1j}, \Omega_{2j}, \Omega_{3j})(p_j) \;=\; \left(\int_{p_j(0)}^{p_j} \omega_1 \,,\, \int_{p_j(0)}^{p_j} \omega_2 \,,\, \int_{p_j(0)}^{p_j} \omega_3 \right).$$

Theorem (Abel)

Suppose that the points p_1, p_2, p_3, p_4 are collinear. Then

$$\Omega_1(p_1) + \Omega_2(p_2) + \Omega_3(p_3) + \Omega_4(p_4) \;=\; 0.$$

The theta surface of $\mathcal{Q}$ is defined by a local parametrization

$$\mathcal{U} \to \mathbb{C}^3,\ (p_1, p_2) \mapsto \Omega_1(p_1) + \Omega_2(p_2).$$

Theorem (Lie)

Given any double translation surface $\mathcal{C}_1 + \mathcal{C}_2 = \mathcal{C}_3 + \mathcal{C}_4$, the arcs $\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2$, $\dot{\mathcal{C}}_3, \dot{\mathcal{C}}_4$ lie on a common quartic $\mathcal{Q}$ in $\mathbb{P}^2$, and $\mathcal{S}$ is the theta surface of $\mathcal{Q}$.

Bernhard Riemann

Suppose that $\mathcal{Q}$ is smooth and let B be its Riemann matrix. This is a complex symmetric 3×3 matrix with positive-definite real part.

The *Riemann theta function* is a holomorphic function in $\mathbf{x} \in \mathbb{C}^3$:

$$\theta(\mathbf{x}, B) \;=\; \sum_{n \in \mathbb{Z}^3} \exp\left(-\pi n^t B n\right) \cdot \cos(2\pi n^t \mathbf{x}),$$

Note: this takes on real values if B and $\mathbf{x}$ are real.

Daniele Agostini and Lynn Chua:

Computing theta functions with Julia, `arXiv:1906.06507`

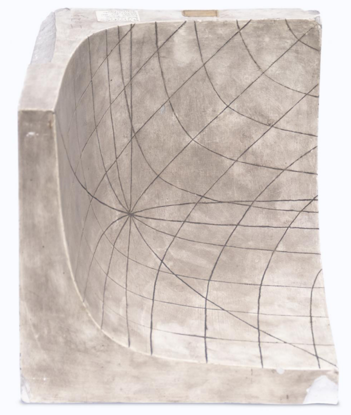
The *theta divisor* is the surface $\Theta_B \;:=\; \{\, \mathbf{x} \in \mathbb{C}^3 \mid \theta(\mathbf{x}, B) = 0 \,\}$.

Theorem (Riemann)

The theta surface $\mathcal{S}$ and the theta divisor Θ_B coincide up to an affine change of coordinates on $\mathbb{C}^3$. We have $\mathcal{S} \;=\; \Pi_\alpha \cdot \Theta_B + c$.

Our Paper

$$\{ 10^{-X} + 10^{-Y} + 10^{-Z} = 1 \}$$



1. Introduction
2. Parametrization by Abelian Integrals
3. Symbolic Computations for Special Quartics
4. Degenerations of Theta Functions
5. Algebraic Theta Surfaces
6. A Numerical Approach for Smooth Quartics
7. Sophus Lie in Leipzig

Symbolic Computation

Algorithm 1: Computing the parametrized theta surface from its plane quartic

Input: The inhomogeneous equation $q(x,y)$ of the plane quartic curve.
Output: The parametrization (8) of the theta surface $\mathcal{S}$ in affine 3-space.
Step 1: Specify two points p_1 and p_2 on the quartic.
Step 2: Fix local parameters (x_1,y_1) and (x_2,y_2) around p_1 and p_2 respectively.
Step 3: Write y_j as an algebraic function in x_j on its branch.
Step 4: Compute the partial derivative q_y on the two branches.
Step 5: Substitute $x_1 = s$ and $x_2 = t$ into the differential forms $\omega_1, \omega_2, \omega_3$ in (5).
Step 6: By integrating these differential forms, compute the vectors

$$\begin{aligned} \Omega_1(p_1(s)) &= \left(\int \frac{s}{q_y(s,y_1(s))}ds, \int \frac{y_1(s)}{q_y(s,y_1(s))}ds, \int \frac{1}{q_y(s,y_1(s))}ds\right), \\ \Omega_2(p_2(t)) &= \left(\int \frac{t}{q_y(t,y_2(t))}dt, \int \frac{y_2(t)}{q_y(t,y_2(t))}dt, \int \frac{1}{q_y(t,y_2(t))}dt\right). \end{aligned}$$

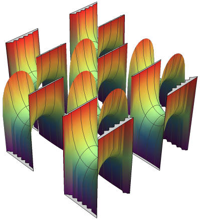
Step 7: Output the sum $\Omega_1(p_1(s)) + \Omega_2(p_2(t))$ of the generating curves.

Input: $q = (y+x-1)(y-x-1)(y+x+1)(y-x+1)$.

Output: $X = \frac{1}{8}(\log(s-1) + \log(t+1)),\ Y = \frac{1}{8}(-\log(s) + \log(t)),$
$Z = \frac{1}{8}(\log(s-1) - \log(s) - \log(t+1) + \log(t))$.

Equation: Setting $u = \exp(8X)$, $v = \exp(8Y)$, $w = \exp(8Z)$, we get
$u^2vw^2 - 2u^2vw - uv^2w + uvw^2 + u^2v - 4uvw - v^2w + uv - uw - 2vw - w = 0$.

Scherk Revisited



Start with $q = xy(x^2 + y^2 + 1)$ and compute derivatives q_x, q_y and differential forms $\omega_1, \omega_2, \omega_3$. The integrals over the line $(s, 0)$ are

$$\begin{aligned}
\Omega_{11}(p_1(s)) &= \int \frac{s}{s(s^2+1)}\,ds &=& \quad \arctan(s),\\
\Omega_{21}(p_1(s)) &= \int \frac{0}{s(s^2+1)}\,ds &=& \quad 0,\\
\Omega_{31}(p_1(s)) &= \int \frac{1}{s(s^2+1)}\,ds &=& \quad \log(s) - \tfrac{1}{2}\log(s^2+1).
\end{aligned}$$

The abelian integral over the line $(0, t)$ are

$$\begin{aligned}
\Omega_{12}(p_2(t)) &= -\int \frac{0}{t(t^2+1)}\,dt &=& \quad 0,\\
\Omega_{22}(p_2(t)) &= -\int \frac{t}{t(t^2+1)}\,dt &=& \quad -\arctan(t),\\
\Omega_{32}(p_2(t)) &= -\int \frac{1}{t(t^2+1)}\,dt &=& \quad -\log(t) + \tfrac{1}{2}\log(t^2+1).
\end{aligned}$$

The Minkowski sum of these two curves is Scherk's minimal surface

$$\sin(X) \;=\; \sin(Y)\cdot\exp(Z).$$

Rational Nodal Quartics

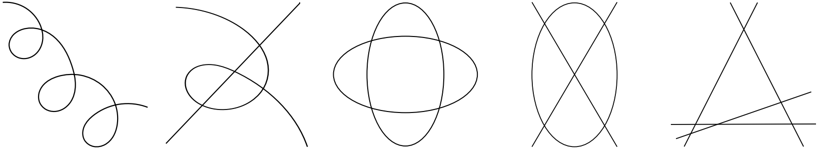


Figure 2: The five types of rational nodal quartics

Let B be the tropical Riemann matrix of the dual graph. We consider a one-parameter family of symmetric 3×3 matrices

$$B_t \; := \; tB + B_0 \qquad \text{for} \quad t \to +\infty.$$

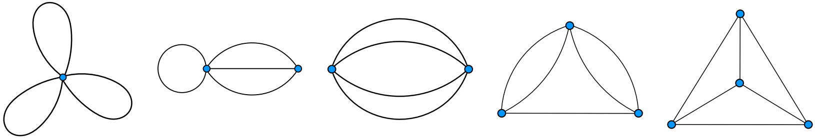
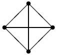
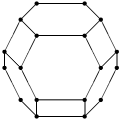
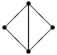
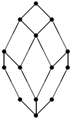
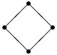
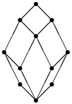
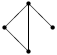
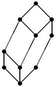
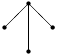
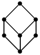


Figure 3: The dual graphs corresponding to the five types of rational nodal quartics

Voronoi Cells

d	Delone Graph	Polytope	Form	Name
6	 K_4		$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$	**TRUNCATED OCTAHEDRON**
5	 $K_4 - 1$		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	**HEXA-RHOMBIC DODECAHEDRON**
4	 C_4		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	**RHOMBIC DODECAHEDRON**
4	 $K_3 + 1$		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	**HEXAGONAL PRISM**
3	 $1+1+1$		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	**CUBE**

Tropical Degeneration

Theorem

Fix a vertex **a** *of the Voronoi cell for the tropical degeneration. The theta surface is defined by the following finite sum over the vertex set* $\mathcal{D}_{\mathbf{a},B}$ *of the Delaunay polytope dual to* **a***:*

$$\sum_{n \in \mathcal{D}_{\mathbf{a},B}} \exp\left(-\pi n^T B_0 n + 2\pi i \cdot n^T \mathbf{x}\right).$$

The number of summands equals 8 *for a rational quartic,* 6 *for a nodal cubic plus line,* 4 *or* 6 *for two conics,* 4 *or* 5 *for a conic plus two lines, and* 4 *for four lines. This recovers Eiesland's formulas.*

$$\{\, 10^{-X} + 10^{-Y} + 10^{-Z} = 1 \,\}$$

B. Bolognese, M. Brandt and L. Chua: *From curves to tropical Jacobians and back*, in Combinatorial Algebraic Geometry, Fields Inst. Comm. **80**, Springer, 2017.

Smooth Quartics

Algorithm 2: Sampling from a theta surface given its plane quartic

Input: The inhomogeneous equation $q(x, y)$ of a smooth plane quartic
Output: A point on the corresponding theta surface in $\mathbb{R}^3$ or $\mathbb{C}^3$
Step 1: Specify two points p_1 and p_2 on the quartic.
Step 2: Take two other points p_1' and p_2' nearby p_1 and p_2 respectively.
Step 3: Compute the following triples of integrals numerically:

$$\begin{aligned} c_1 &= \left(\int_{p_1}^{p_1'} \omega_1, \int_{p_1}^{p_1'} \omega_2, \int_{p_1}^{p_1'} \omega_3\right), \\ c_2 &= \left(\int_{p_2}^{p_2'} \omega_1, \int_{p_2}^{p_2'} \omega_2, \int_{p_2}^{p_2'} \omega_3\right). \end{aligned}$$

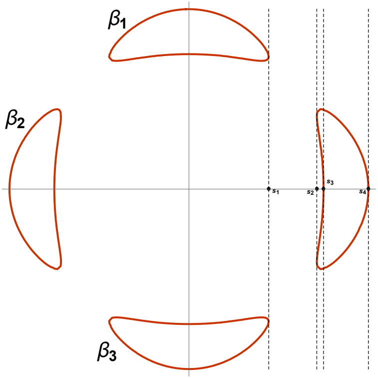
Step 4: Output the sum $c_1 + c_2$.

Our engine for Step 3 is the **Sage** package `RiemannSurfaces` from

N. Bruin, J. Sijsling and A. Zotine: *Numerical computation of endomorphism rings of Jacobians*, Proceedings 13th Algorithmic Number Theory Symposium (ANTS, 2019)

The Trott Curve

$$q = 144(x^4 + y^4) - 225(x^2 + y^2) + 350x^2y^2 + 81$$



$$B = \begin{pmatrix} 0.926246 & 0.553994 & 0.372252 \\ 0.553994 & 1.10799 & 0.553994 \\ 0.372252 & 0.553994 & 0.926246 \end{pmatrix}$$

Algebraic Theta Surfaces

Theorem (Eiesland)

Every algebraic theta surface is rational and has degree $3, 4, 5$ *or* 6*. Its quartic has rational components and no singularities are nodes.*

Example

Consider the toric curve $x^4 - yz^3$ in $\mathbb{P}^2$. This rational quartic has a triple point. The abelian integrals evaluate to

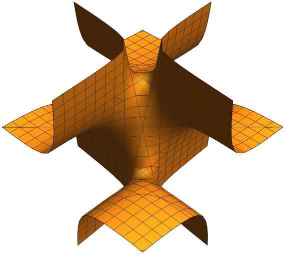
$$\begin{aligned} X &= \int s ds + \int t dt &= \tfrac{1}{2}(s^2 + t^2), \\ Y &= \int s^4 ds + \int t^4 dt &= \tfrac{1}{5}(s^5 + t^5), \\ Z &= \int ds + \int dt &= s + t. \end{aligned}$$

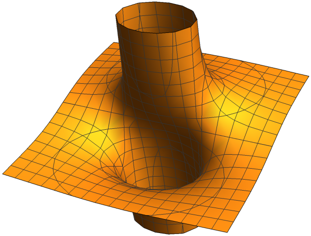
Elimination of the parameters s and t reveals the theta surface

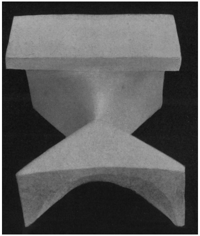
$$Z^5 - 20X^2 Z + 20Y = 0.$$

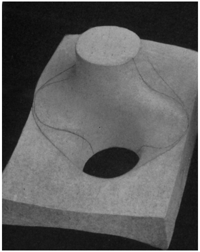
John Little: *Translation manifolds and the converse to Abel's theorem* Compositio Mathematica (1983)

The Cardiod Surface and The Deltoid Surface









$8X^2Z + 8Y^2Z - 4YZ - X$ $\quad$ $4XYZ + 4XY + 2XZ - 2YZ + 3X - Y$

Integrable Systems

We derived the quintic $Z^5 - 20X^2Z + 20Y$ from $x^4 - yz^3$.

This is a Schur-Weierstrass polynomial, arising from a *degeneration of the sigma function* for $(3,4)$-curves.

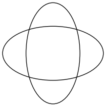
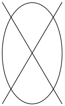
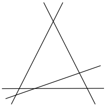
V.M. Buchstaber, V.Z. Enolski and D.V. Leykin: *Rational analogs of abelian functions*, Functional Analysis and its Applications (1999).

A. Nakayashiki: *Tau function approach to theta functions*, International Mathematics Research Notices (2016).

Plan: Apply our tools to the **KP equation** for shallow water waves

B. Dubrovin, R. Flickinger and H. Segur: *Three-phase solutions of the Kadomtsev-Petviashvili equation*, Studies in Applied Math (1997).

Sophus Lie and Tropical Geometry

Already in 1869, Lie had shown how to construct infinitely many representations $\mathcal{S} = \mathcal{C}_1 + \mathcal{C}_2$ for the *tetrahedral theta surface*

$$\mathcal{S} \;=\; \{\, \alpha \cdot \exp(X) + \beta \cdot \exp(Y) + \gamma \cdot \exp(Z) \;=\; \delta \,\}.$$

Lie introduces coordinates $X = \log(U),\; Y = \log(V),\; Z = \log(W)$ and he studies *Hadamard product decompositions* $\mathcal{P} = \mathcal{D}_1 \cdot \mathcal{D}_2$ of

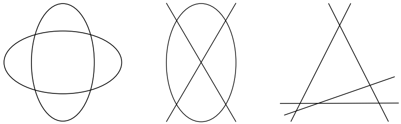
$$\mathcal{P} \;=\; \{\, \alpha \cdot U + \beta \cdot V \;+\; \gamma \cdot W \;=\; \delta \,\}.$$

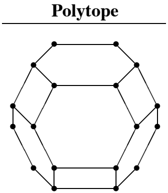
View this plane in $\mathbb{P}^3$ and translate it to contain $\mathbf{1} = (1:1:1:1)$. Intersecting $\mathcal{P}$ with the coordinate planes gives lines $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$.

Theorem (Lie)

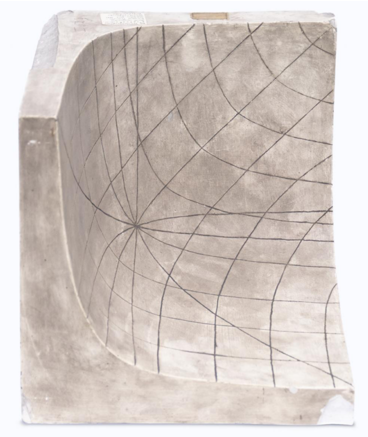
Let $\mathcal{D}_1, \mathcal{D}_2$ be lines through $\mathbf{1}$ in $\mathcal{P}$. Then $\mathcal{P} = \mathcal{D}_1 \cdot \mathcal{D}_2$ if and only if the six lines $\mathcal{D}_i\,, \mathcal{L}_j$ are tangent to a common conic in $\mathcal{P}$.

Sophus Lie and Amoebas in 1869

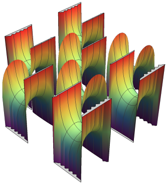




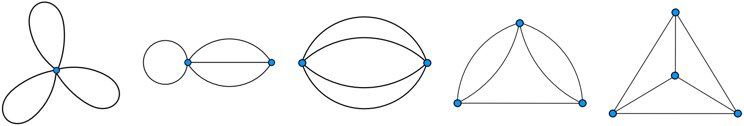
$$\mathcal{S} \;=\; \{\, \alpha \cdot \exp(X) + \beta \cdot \exp(Y) + \gamma \cdot \exp(Z) = \delta \,\}$$



Forty Years Earlier



Niels Henrik Abel visited Freiberg (also in Saxony) in **1825-1826**



From Wikipedia:

"…. From Berlin Abel also followed his friends to the Alps. He went to Leipzig and Freiberg to visit Georg Naumann and his brother the mathematician August Naumann.

In Freiberg Abel did research in the theory of functions, particularly, elliptic, hyperelliptic, and a new class now known as ***abelian functions****.*

Thank You