ALGEBRAIC GEOMETRY AND REPRESENTATION THEORY IN THE VERLINDE CATEGORY

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1. Symmetric Tensor Categories

I want to begin by introducing the general setting of symmetric tensor categories in a relatively non-technical way. Let us fix some algebraically closed field $\mathbf{k}$

Definition 1.1. A *symmetric tensor category* over $\mathbf{k}$ is a category $\mathcal{C}$ equipped with the following extra structure:

1. $\mathcal{C}$ is $\mathbf{k}$-linear and locally finite as an abelian category.

2. $\mathcal{C}$ is equipped with a bilinear *tensor product* bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Think of this as a multiplication internal to $\mathcal{C}$.

3. This multiplication is associative and there is an object $\mathbf{1}$ that is unital for this multiplication. In general, associativity and unitality in the categorical setting can be complicated but for this talk, you can simply read associativity as allowing you to ignore parentheses, and unitality to mean

$$\mathbf{1} \otimes X = X = X \otimes \mathbf{1}, \text{ for any } X \in \mathcal{C}.$$

4. Every object $X \in \mathcal{C}$ has a dual X^* that comes equipped with a pair of maps

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}, \ \mathrm{coev}_X : \mathbf{1} \to X \otimes X^*.$$

   These maps satisfy a relation that is somewhat tedious to write down but is very easy to understand intuitively. Acting via ev_X allows us to view $X \otimes X^*$ as matrices on X, and the relation then says that coev_X must be the inclusion of the identity matrix.

5. The above data defines a structure of a tensor category on $\mathcal{C}$. For this to be a *symmetric* tensor category, we need a notion of swapping tensors, namely a natural isomorphism

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

   for every pair of objects $X, Y \in \mathcal{C}$ that squares to the identity.

All this structure needs to satisfy a bunch of extra compatibility relations but describing this in detail is not illuminating at all. So, instead, let us look at some examples.

Example 1.2. 1. The simplest symmetric tensor category is the category of finite dimensional vector spaces, with $\otimes$ being tensor product over the field. The unit object is the ground field and c is just the swap map. This is the initial symmetric tensor category, in the sense that it sits inside every symmetric tensor category as the subcategory generated by $\mathbf{1}$.

2. More generally, if H is a commutative Hopf algebra over $\mathbf{k}$, then the category of finite dimensional H-comodules forms a symmetric tensor category, with essentially the same structure as the previous example.

3. Here is a slightly different example. If $char(\mathbf{k}) \neq 2$, then the category of finite dimensional super vector spaces is a symmetric tensor category. As a tensor category, it is equivalent to $\mathrm{Rep}_{\mathbf{k}}(\mathbb{Z}/2\mathbb{Z})$, but the symmetric structure comes from the signed swap map, rather than the swap map, where permuting two odd elements picks up a -1.

4. In analogy with the Hopf algebra example, the category of finite dimensional comodules of a supercommutative Hopf super algebra H is also a symmetric tensor category, using the signed swap map. I will use the term "representations of a super group scheme" to refer to such categories, thinking of $\mathrm{Spec}(H)$ as an affine supergroup.

So why should anyone care about symmetric tensor categories? To me, there are two main sources of motivation:

1. Representation Theory: As the above examples show, symmetric tensor categories are a good setting in which to study the representations of algebraic groups or supergroups.

2. Algebraic Geometry: The structure of a symmetric tensor category is the minimal structure needed to be able to define commutative algebras. The geometric properties of commutative algebras in symmetric tensor categories is very interesting.

In characteristic zero, it turns out that symmetric tensor categories aren't much more general than the last of the examples I provided. More precisely, there is the following theorem of Deligne.

Theorem 1.3 (Deligne)**.** Let $\mathcal{C}$ be a symmetric tensor category over an algebraically closed field of characteristic 0 and assume that for each object $X \in \mathcal{C}$, there is a constant a_X such that

$$length(X^{\otimes n}) \leq a_X^n.$$

Then, $\mathcal{C}$ is the category of finite dimensional representations of an affine supergroup.

Hence, symmetric tensor categories in characteristic 0 are either large, and hence tend to have bad Noetherianity properties that makes it difficult to do much algebraic geometry, or they are simply representations of algebraic groups and Lie superalgebras. In characteristic $p > 0$ however, this theorem fails to be true. For most of the rest of this talk, I want to explain how much of this we can replicate in characteristic p, and what new phenomena appear.

2. The Verlinde Category

For the rest of this talk, fix $p > 0$ as the characteristic of $\mathbf{k}$. The simplest counterexample to Deligne's theorem, at least when $p \geq 5$ is the Verlinde category Ver_p associated to SL_2. I will first describe this category in detail, and then show how to see it is a counterexample, and also give some more fundamental reasons why you should care about it. The main construction involved is semisimplification.

Let $\mathcal{C}$ be a symmetric tensor category, not necessarily semisimple. For any object $X \in \mathcal{C}$ and any endomorphism $\rho : X \to X$, there is a notion of trace $\mathrm{Tr}(\rho) \in \mathbf{k}$. I won't go into the full details of how this is defined, but as an example, if $\mathcal{C}$ is the category of representations of a group scheme, then this is the ordinary trace, and if $\mathcal{C}$ is the category of representations of a super group scheme, this is the super trace. To *semisimplify* $\mathcal{C}$, we now take the following steps:

1. For $f : X \to Y$ in $\mathcal{C}$, we say f is *negligble* if for any $g : Y \to X$, $g \circ f$ is traceless.

2. Define $\mathcal{N}$ to be the ideal in $\mathcal{C}$ consisting of the same objects but with $\mathcal{N}(X, Y)$ the space of neglgible morphisms. This is a tensor ideal, i.e., it is closed under addition, scalar multiplication, composition and tensor product.

3. The semisimplification $\overline{\mathcal{C}}$ is $\mathcal{C}/\mathcal{N}$. It is a semisimple symmetric tensor category with a simple object for each indecomposable in $\mathcal{C}$ for which the trace of the identity is not 0.

4. Ver_p is obtained by semisimplifying any one of three categories: representations of $\mathbb{Z}/p\mathbb{Z}$, representations of α_p (the first Frobenius kernel of $\mathbb{G}_a$) or the category of tilting modules for SL_2.

So what does Ver_p look like. Well the multiple ways to construct it allow us to very easily describe its additive and monoidal structure.

1. As an abelian category, $\mathrm{Rep}(\mathbb{Z}/p\mathbb{Z})$ has p-indecomposables corresponding to the Jordan blocks of size 1 through p. The only indecomposable of dimension p is the free module. Hence, Ver_p has $p-1$ simples, $V_0, \dots V_{p-2}$, with the index corresponding to the dimension of the $\mathbb{Z}/p\mathbb{Z}$ representation.

2. The tensor products of simples in Ver_p follow a truncated version of the tensor product rules in $\mathrm{Rep}(SL_2, \mathbb{C})$. In $\mathrm{Rep}(SL_2, \mathbb{C})$,

$$V_i \otimes V_j = V_{i-j} \oplus V_{i-j+2} \oplus \cdots \oplus V_{i+j}.$$

   If $i+j < p-1$, then the tensor product is the same in Ver_p. But if $i+j = p-1+n$, then to get the tensor product in Ver_p, start with the above decomposition and delete everything between V_{p-1-n} and V_{p-1+n}.

2.1. **Motivation:**

It's very easy to see from the tensor product rules that only V_0 and V_{p-2} could have integer dimension. Take the case $p = 5$ for example, here

$$V_2^{\otimes 2} = V_2 + V_0$$

and hence its dimension is the golden ratio. As a result, Ver_p for $p \geq 5$ cannot be the category of representations of a supergroup, and is a counterexample to Deligne's theorem. But if this were the only thing interesting, then the category would just be a curiosity. My interest in Ver_p stems from three important facts:

1. **Universailty:** Ver_p is a partial replacement for sVec in Deligne's theorem. Namely, there is a theorem of Victor Ostrik that says that any symmetric fusion (finite + semisimple tensor) category over $\mathbf{k}$ is the category of comodules of some commutative Hopf algebra in Ver_p.

2. **Connections to Modular Representation Theory:** The three different ways to construct Ver_p give us a lot of ways to construct interesting algebras, Hopf algebras and Lie algebras in Ver_p from corresponding objects in $\mathrm{Rep}(\mathbb{Z}/p\mathbb{Z}), \mathrm{Rep}(\alpha_p)$ or $Tilt(SL_2)$. Additionally, Vec and sVec sit as tensor summands of Ver_p, so from these objects we can obtain ordinary algebras, Hopf algebras and Lie algebras (along with super versions). Hence, by passing through Ver_p, we can give fairly elementary constructions of interesting objects in modular representation theory.

   For example, if $p = 5$, we can take the Lie algebra $\mathfrak{g} = E_8$ and a principal nilpotent element N coming of an $\mathfrak{so}(5)$ subalgebra. The action of $[N, -]$ on $\mathfrak{g}$ turns $\mathfrak{g}$ into a Lie algebra in $\mathrm{Rep}(\alpha_p)$ and we can semisimplify to get a Lie algebra $\overline{\mathfrak{g}}$ in Ver_p. Projecting down to sVec gives us the Elduque exceptional Lie superalgebra, of which there isn't really any other elementary construction. This relationship between Ver_p and modular representation theory is still very unexplored and interesting.

3. **Connections to Modular Tensor Categories in Characteristic 0:** There seems to be a deep relationship between symmmetric fusion categories in positive characteristic and certain modular tensor categories in characteristic 0 associated to quantum groups at roots of unity that is controlled by semisimple Lie algebras in Ver_p. Time permitting, I will describe some aspects of this connection at the end of the talk, as it relates to a very interesting and wide open conjecture of Victor Ostrik regarding classification of symmetric fusion categories.

Hopefully, I've managed to convince you that the Verlinde category is worth studying. For the rest of this talk, I will use Ostrik's theorem as motivation to show some important properties of commutative Hopf algebras in Ver_p. The main goal will be to show that the data of a commutative Hopf algebra in Ver_p is the same as the data of a Hopf algebra over $\mathbf{k}$, a Lie algebra in Ver_p along with some compatibility of adjoint actions. This will reduce the extra complexity of going from vector spaces to Ver_p to a purely local problem of understanding Lie algebras in Ver_p and their representations.

3. **Commutative Algebra in** Ver_p

In order to prove this result, the main tool is splitting a commutative Hopf algebra in Ver_p as a direct product of an underlying commutative Hopf algebra over $\mathbf{k}$ and an algebra of functions on its Lie algebra. To do so, I need to explain some basic features of commutative algebras in Ver_p. However, commutative algebras in Ver_p itself form a very limited class, they are analogous to the Artinian commutative $\mathbf{k}$-algebras. Hence, when I speak about commutative algebras in Ver_p, I will actually allow them to have infinite length. In this bigger class of algebras, to get reasonable geometric properties, we need some sort of finiteness condition. The correct notion turns out to be *finite generation*.

Definition 3.1. A commutative algebra in $\mathrm{Ver}_p^{\mathrm{ind}}$ is *finitely generated* if it is the quotient of $S(X)$ for some object $X \in \mathrm{Ver}_p$.

Here $S(X)$ is the symmetric algebra of X, i.e., the coinvariants of the symmetric braiding in the tensor algebra of X.

Finitely generated commutative algebras in Ver_p turn out to be very nice geometrically, and their good behavior stems from one key result.

Theorem 3.2 (V.)**.** If L is any simple object in Ver_p not isomorphic to $\mathbf{1}$, then

$$S^n(L) = 0 \text{ for } n \geq p-1.$$

Proof. The proof of this theorem relies on the relationship between Ver_p and $\mathrm{Ver}_p(SL_n)$ for $2 < n < p$. Let $L = V_i$ for $i > 1$. Then, we can construct a Verlinde category $\mathrm{Ver}_p(SL_{i+1})$ by semisimplifying the category of tilting modules for SL_{i+1}. Restriction to the principal nilpotent gives us a symmetric tensor functor

$$F : \mathrm{Ver}_p(SL_{i+1}) \to \mathrm{Ver}_p$$

that sends the tautological representation V to L. But in $\mathrm{Ver}_p(SL_{i+1})$ it is immediate that $S^{p-1}(V) = 0$, as it corresponds to a negligible indecomposable tilting module, i.e., a tilting module that is killed by semisimplification.

□

This has some important consequences.

Corollary 3.3. Let A be a finitely generated commutative algebra in Ver_p.

1. If I is the ideal generated by all simple subobjects of A not isomorphic to $\mathbf{1}$, then I is in the kernel of the Frobenius endomorphism on A. In particular, I is nilpotent.
2. A is Noetherian.
3. The invariant subalgebra $A^{\mathrm{inv}} = Hom(\mathbf{1}, A)$ is a finitely generated commutative $\mathbf{k}$-algebra and A is a finite extension of A^{inv}. Hence, A is both a finite extension and a nilpotent thickening of finitely generated commutative $\mathbf{k}$-algebras.

These facts largely trivialize commutative algebra in Ver_p. Many basic theorems that you would like to prove such as Nullstellensatz or Krull Intersection Theorem follow immediately from this corollary. This result also shows that topologically, commutative algebras in Ver_p are no different from commutative

$\mathbf{k}$-algebras, and so we can even define non-affine schemes in Ver_p by gluing along principal open affines in A^{inv}.

However, it is important to note that the space of gluing isomorphisms is now very different, as these morphisms now must live in Ver_p. Hence, non-affine schemes still have a lot of interesting questions open to study. One that I'm thinking about at the moment for example, is the cohomology of "line" bundles associated to flag varieties associated to simple objects not isomorphic to $\mathbf{1}$, as this has implications on Borel-Weil-Bott theory in Ver_p. But this is all out of the scope of this talk. For today, what matters is that the above result gives a lot of control on the geometric properties of finitely generated commutative Hopf algebras in Ver_p.

4. **Algebraic Groups in Ver_p and Harish-Chandra pairs**

Let us now return to our main focus of trying to understand commutative Hopf algebras in Ver_p. Ver_2 is just the category of vector spaces and Ver_3 is just the category of super vector spaces so let us assume for this section that $p > 3$. We will also assume that all commutative algebras are finitely generated here. I'm now going to introduce some constructions and terminology.

1. Given a commutative Hopf algebra H in Ver_p, I will call $G = \mathrm{Spec}(H)$ the affine group scheme associated to H, even though $\mathrm{Spec}(H)$ isn't really a precise term. G is really just H itself (or more precisely, the group valued functor $Hom(H, -)$ on the category of commutative algebras in Ver_p) but the scheme theoretic terminology is useful. I will also use $\mathcal{O}(G)$ to mean the Hopf algebra H that defined G and call it the algebra of functions on G.

2. If I is the ideal generated by simple summands of H that are not isomorphic to $\mathbf{1}$, then I is a Hopf ideal by semisimplicity of Ver_p. Hence, $\overline{H} = H/I$ is a commutative Hopf algebra over $\mathbf{k}$. I will use G_0 to denote $\mathrm{Spec}(\overline{H})$ and call it the underlying ordinary group scheme.

3. The *distribution algebra* of G, or the *dual coalgebra* of $\mathcal{O}(G)$, is the cocommutative coalgebra
$$\mathcal{O}(G)^\circ := \bigcup_n (\mathcal{O}(G)/J^n)*,$$
where J is the augmentation ideal, i.e., the kernel of the counit map on $\mathcal{O}(G)$.

4. The *Lie algebra* associated to G is
$$Lie(G) := (J/J^2)^*.$$
It is a Lie algebra in Ver_p, as it is the space of primitives inside $\mathcal{O}(G)^\circ$.

5. For a Lie algebra $\mathfrak{g}$ in Ver_p, the subobject $\mathfrak{g}_0$ generated by the summands isomorphic to $\mathbf{1}$ is a $\mathbf{k}$-Lie algebra which I will call the ordinary Lie subalgebra. If $\mathfrak{g} = Lie(G)$, it is easy to see that $\mathfrak{g}_0 = Lie(G_0)$.

These are a lot of definitions. Let me show you how it works with one important example.

Example 4.1. If X is an object in Ver_p, then $G = GL(X)$ is an affine group scheme in Ver_p. There are two ways to describe G. First, via the algebra of functions,

$$\mathcal{O}(G) = \mathrm{Sym}[(X \otimes X^*)^{\oplus 2}]/I$$

where I is the ideal cutting out the relations $AB = BA = 1$, if we think of A and B as elements of $X \otimes X^*$ with multiplication given by evaluation in the middle. This is not very precise, but writing down I explicitly in terms of the multiplication map and coevaluation is not hard, but very tedious.

Alternatively, we can view G through the functor of points

$$G(A) = Hom(\mathcal{O}(G), A) = \text{ A-module automorphisms of } A \otimes X,$$

for any commutative algebra A in Ver_p.

Let us see how the constructions in the above definition work for $G = GL(X)$.

1. If
$$X = \bigoplus_{i=0}^{p-2} n_i V_i,$$
then
$$G_0 = \prod_i GL(n_i, \mathbf{k}).$$
2. $$Lie(G) = \mathfrak{gl}(X) = X \otimes X^*$$
where the Lie bracket is just the commutator, as $X \otimes X^*$ is an associative algebra under middle evaluation
$$X \otimes X^* \otimes X \otimes X^*.$$

These constructions show that given an affine group scheme G in Ver_p, we can produce the following data:

1. An affine group scheme G_0 over $\mathbf{k}$.
2. A Lie algebra $\mathfrak{g} = Lie(G)$ in Ver_p such that $\mathfrak{g}_0 = Lie(G_0)$.
3. An action of G_0 compatible with the Lie algebra structure.
4. The action of $\mathfrak{g}_0$ on $\mathfrak{g}$ obtained via differentiating the action of G_0 coincides with the action obtained by restricting the adjoint action of $\mathfrak{g}$ on itself.

If we now forget that the group G exists, and just take the above structure and compatibility conditions, then we have a *Harish-Chandra pair* in Ver_p. Harish-Chandra pairs naturallt form a category in the obvious way, and our constructions give us a functor from the category of affine group schemes in Ver_p to the category of Harish-Chandra pairs in Ver_p. The main theorem of this talk is the following.

Theorem 4.2 (V.)**.** This functor establishes an equivalence between the category of affine group schemes in Ver_p and the category of Harish-Chandra pairs in Ver_p.

Proof. I am not going to give a full proof of this theorem, it is fairly technical. Instead, I will simply describe the construction of the inverse functor and then give a rough idea of what goes into proving that the construction is truly an inverse.

To construct an inverse, what I need to do is start with a Harish-Chandra pair and produce an affine group scheme in Ver_p. So let us suppose we have a Harish-Chandra pair $(G_0, \mathfrak{g})$. Rather than constructing the group G directly, it is easier to construct the dual coalgebra $\mathcal{O}(G)^\circ$.

Decompose
$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\neq 0}$$
in Ver_p. The tensor algebra $T(\mathfrak{g}_{\neq 0})$ is a Hopf algebra with $\mathfrak{g}_{\neq 0}$ primitive and is equipped with a Hopf action of $\mathcal{O}(G_0)^\circ$. So, we can take the smash product
$$\mathcal{H}(G_0, \mathfrak{g}) = \mathcal{O}(G_0)^\circ \ltimes T(\mathfrak{g}_{\neq 0}).$$
This doesn't encode the commutation relations in $\mathfrak{g}$. So we define I to be the ideal generated by

$$X \otimes Y - Y \otimes X - [X, Y]$$

for $X, Y \in \mathfrak{g}_{\neq 0}$. This can be done without any reference to elements of $\mathfrak{g}_{\neq 0}$, and while $[X, Y]$ doesn't have to land in $\mathfrak{g}_{\neq 0}$, this is fine, because $\mathfrak{g}_0 \subseteq \mathcal{O}(G_0)^{\circ}$.

On the commutative side, we have $\mathcal{O}(G_0)$ coacting on the tensor coalgebra $T_c(\mathfrak{g}^*_{\neq 0})$ (the graded dual of $T(\mathfrak{g}_{\neq 0})$). Hence, we can take the smash product

$$\mathcal{A}(G_0, \mathfrak{g}) = \mathcal{O}(G_0) \ltimes T_c(\mathfrak{g}^*_{\neq 0})$$

and we can define $\widehat{\mathcal{A}}(G_0, \mathfrak{g})$ to be its completion with respect to the grading on $T_c(\mathfrak{g}^*_{\neq 0})$. This has a perfect pairing with $\mathcal{H}(G_0, \mathfrak{g})$, and the inverse functor A is defined by

$$A(G_0, \mathfrak{g}) = I^{\perp}$$

under this pairing.

The proof that this is an inverse construction builds on work by Akira Masuoka, who proved the analogous result for the category of supervector spaces, and the two key ideas in extending his work are the PBW theorem for Lie algebras in Ver_p proved by Pavel Etingof, and the fact that the Frobenius map is trivial on $\mathfrak{g}_{\neq 0}$. This last fact ensures that in the dual coalgebra, divided powers only show up for $\mathfrak{g}_0$, which is used to show that $\mathcal{H}(G_0, \mathfrak{g})/I$ is the correct dual coalgebra, and then we can use the perfect pairing to move the result over to $A(G_0, \mathfrak{g})$.

□

So why is this theorem useful? Well, Harish-Chandra pairs turn out to be easier to work with for the most part than affine group schemes in Ver_p, the main reason being that Lie algebras are local data and have finite length in Ver_p, versus the typically infinite length Hopf algebras defining the group schemes. To end this talk, I will show a few applications of this result.

5. **Applications**

5.1. **Representation Theory.** To me, the interesting thing isn't necessarily the affine group schemes themselves but their categories of representations, because these give us a large class of examples of symmetric tensor categories in positive characteristic.

Definition 5.1. If G is an affine group scheme in Ver_p, then a representation V for G is a homomorphism of groups $G \to GL(V)$.

Definition 5.2. If $\mathfrak{g}$ is a Lie algebra in Ver_p, then a representation V of $\mathfrak{g}$ is a homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V) = V \otimes V^*$.

As an easy consequence of the main theorem, we have

Corollary 5.3. If G is an affine group scheme in Ver_p, then a representation of G is the same thing as a representation of $\mathfrak{g}$ such that the action of $\mathfrak{g}_0$ lifts to G_0.

In practice, this latter condition ends up being much easier to check.

5.2. **Irreducible Representations of $GL(X)$.** An interesting problem is to classify the irreducible representations of $GL(X)$ for X an object in Ver_p. If

$$X = \bigoplus n_i V_i$$

then parabolic induction shows that the irreducibles are in bijection with the irreducibles of

$$\prod_i GL(n_i V_i).$$

Hence , we can assume that X has a single isotypic component.

To classify the irreducibles of $GL(n_i V_i)$, the first step is to understand what happens when $n_i = 1$. In this case, $G_0 = GL(1)$ is a central subgroup, so define $PGL(V_i) = G/G_0$. At the level of Lie algebras, we get a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{sl}(V_i)$$

where $\mathfrak{sl}(V_i)$ is the kernel of the evaluation map ϵ. Hence, we get a homomorphism of Harish-Chandra pairs $(1, \mathfrak{sl}(V_i)) \to (G_0, \mathfrak{g})$. By the equivalence between Harish-Chandra pairs and groups, this means that the quotient map $G \to PGL(V_i)$ splits and hence

$$G = GL(1) \times PGL(V_i).$$

Hence,

$$\mathrm{Rep}(GL(V_i)) = \mathrm{Rep}(GL(1)) \times \mathrm{Rep}(PGL(V_i))$$

and $\mathrm{Rep}(PGL(V_i))$ turns out to be $\mathrm{Ver}_p^+(SL_{i+1})$, a connected subcategory of $\mathrm{Ver}_p(SL_{i+1})$.

For $n > 1$, we can think of $GL(nV_i)$ as the group of $n \times n$ matrices with entries in $\mathfrak{gl}(V_i)$ with nonzero determinant. As such, we have an obvious triangular decomposition into diagonal, upper triangular and lower triangular matrices, and the maximal torus is

$$GL(V_i)^n.$$

Hence, $GL(nV_i)$ seems to have a highest weight theory, with irreducibles labelled by certain dominant and integral weights, where the weights are pairs of weights for GL_n and simple objects in $\mathrm{Ver}_p^+(SL_{i+1})$. However, dominance and integrality is still an open question, it doesn't seem to be the same as taking dominant integral weights for $GL(n)$.

5.3. Classification of symmetric fusion categories.

The last thing I will talk about is a problem of classification. Recall that Ostrik's theorem stated that any symmetric fusion (finite, semisimple, tensor) category was the category of representations of some commutative Hopf algebra (now of finite length) in Ver_p. Let G be the associated affine group scheme. Then, since $\mathcal{O}(G)$ has finite length, looking at the corresponding Harish-Chandra pair, we must have $\mathfrak{g}_{\neq 0}$ is a Lie algebra as well and

$$\mathcal{O}(G) = \mathcal{O}(G_0) \ltimes U(\mathfrak{g}_{\neq 0})^*$$

with $\mathcal{O}(G_0)$ a semisimple group scheme of finite dimension over $\mathbf{k}$. So, to classify the symmetric fusion categories over $\mathbf{k}$, we need to classify all such Lie algebras with semisimple representation theory. So far, every example of these Lie algebras that we have constructed seem to have Verlinde categories associated to other algebraic groups as their representation categories. This led to a big open conjecture by Ostrik

Conjecture 5.4. Every symmetric fusion category in positive characteristic is the product of a pointed category with the equivariantization of a Verlinde category associated to an algebraic group.

If this conjecture were true, then it would imply a deep connection between modular tensor categories associated to quantum groups in characteristic 0 and symmetric fusion categories in characteristic p, because these Verlinde categories are reductions mod p of the semisimplification of quantum group representations at roots of unity.