I - Some Homological algebra

\[ \text{Ch}^b(\mathcal{A}) \xrightarrow{\text{mod homotopy}} K^b(\mathcal{A}) \xrightarrow{\text{invt. quasi-iso}} \mathcal{D}^b(\mathcal{A}) \]

\[ K^b(\text{Proj} \mathcal{A}) \xrightarrow{\text{fully faithful}} \]

Sometimes an equivalence (with finite projective dimension)

Even when \( K^b(\text{Proj} \mathcal{A}) = \mathcal{D}^b(\mathcal{A}) \), the latter is better for some applications, e.g., rt. der. factors.

\[ \text{Theme} \rightarrow \text{always fully faithful, sometimes better} \]

Next example: DG algebra

Let \( \mathcal{B} \) be a dg-alg/\( K \). This means:

1. \( \mathcal{B} \) is a chain comp. of v.s. \( \mathcal{B}^i \xrightarrow{d} \mathcal{B}^i \xrightarrow{d} \mathcal{B}^j \)
2. Mult. map \( m: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B} \) that is a chain map, i.e., \( d \) obeys Leibniz rule
3. \( (\mathcal{B}, m) \) is a graded algebra

A dg-module \( \mathcal{M} \) over \( \mathcal{B} \) consists of

1. Chain comp. \( \mathcal{M} \).
2. Mult. map \( m: \mathcal{B} \otimes \mathcal{M} \rightarrow \mathcal{M} \) that is a chain map.

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2. Multi. map \( m : \mathcal{B} \otimes M \to M \) that is a chain map.
3. Map from 2 makes \( M \) into a \( \mathcal{B} \)-module.

\[ \mathcal{B}-\text{mod}_{\mathcal{B}} \xrightarrow{\text{mod by } \mathcal{B}} \text{Hdg}^{0} \xrightarrow{\text{inv. part}} \text{Dg}^{0} \]

\[ \text{cat}_{\mathcal{B}} \otimes \text{dg-mod}_{/\mathcal{B}} \xrightarrow{\text{Hdg perf}} \text{Dg}^{0} \]

Perfect = underlying module is free of finite rank / \( \mathcal{B} \)

\[ \text{Hdg perf}^{0} \xrightarrow{\text{fit into the theme}} \text{Dg}^{0} \]

Aoo-module \( M \) over a dg-dg \( \mathcal{B} \) consists of:
1. Chain cplx. \( M \).
2. Chain map \( \mathcal{B} \otimes M \to M \).
3. The map \( \mathcal{B} \otimes \mathcal{B} \otimes M \to M \) defined by a specified homotopy.
4. Choice of homotopy \( \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \to M \) is governed by specified higher homotopy.
5. Keep going...

Aoo-chain maps \( \mathcal{E} \) are coming from defn of "mg/mult:" Aoo-homotopies I only imposed up to homotopy.

\[ \mathcal{B}-\text{mod}_{\mathcal{B}} \xrightarrow{\text{Aoo-hom}} \text{Hoo}^{0} \xrightarrow{\text{inv. part}} \text{Doo}^{0} \]

Miracle: Under mild assumptions on \( \mathcal{B} \), giss are already inv. up to homotopy and
We've seen why $K_\ast P$ is important

$$H = \text{Sym}(V), \text{deg } V = 2.$$
\[ H = \text{Sym}(V), \quad \text{deg } V = 2, \quad \text{a v.g. that act on graded Hom spaces in } \mathcal{P} \]

\[ \mathcal{P} \]

e.g. could take

1. \( H = H^* \)

2. \( H = \text{Sym}(1^*) \circ \text{Sym}(1) = \text{Sym}(1 \otimes 1) \)

3. \( H = \text{Sym}(1^*) \circ \text{Sym}(1) = \text{Sym}(1 \otimes 1) \)

\[ \text{Problem: Replace } K^b \mathcal{P} \text{ by a new thing cat-\( y \) in which the action of } H \text{ is trivial} \]

Think: \( K^b \mathcal{P} \)

e.g.

1. Forget equivalence

2. Kill or copy of \( \text{Sym}(1^*) \sim \text{Siegel module} \)

3. Approach to the problem

1. "Naive" Let \( \bar{\mathcal{P}} = \text{cat-\( y \) with same objects as } \mathcal{P} \)

morphism moded by \( K^b \mathcal{P}(1) \).

Work with \( K^b \bar{\mathcal{P}} \).

Only work if Hom in \( \mathcal{P} \) were free/\( H \) ("H-free situation")

2. "deg" Replace \( \mathcal{P} \) by a deg-deg g-iso to \( \mathcal{P} \text{ flat}/\mathcal{H} \)
4) "dg" Replace $\mathcal{D}$ by a dg-$\mathcal{D}$-qcqs to $k$, flat/H

eq \text{let}
\mathcal{B} = H \otimes_{\Delta V} \text{equipped with some differential}

(e.g. Koszul complex of $k/H$)

$\text{Ho}_{dg, perf}(\mathcal{B}, \mathcal{P})$

Objects: Pairs $(\mathcal{F}, \theta)$

- ordinary chain comp of obj. in $\mathcal{P}$.
- $\mathcal{F} : \mathcal{B} \to \text{End}(\mathcal{F})$ a dg-$\mathcal{D}$ hom. at $\text{End}(\mathcal{F})$

in $\text{free}/\mathcal{B}$

3) "A∞" $\text{Ho}_{A∞}(\mathcal{B}, \mathcal{P})$

Objects: $(\mathcal{F}, \theta)$

$\mathcal{F}$ as before,

$\theta$ an $A∞$-dg hom (btw. dg. dg)

E.g. 2) $\mathcal{P} = \mathcal{C}^*$-eq. parity sh. on $\mathcal{C}^*$

$K^b\mathcal{P} = D^b(\mathcal{C}^*)$

$\text{Ho}_{dg, perf}(\mathcal{B}, \mathcal{P})$ contains loc. system w/ exp.

monodromy

$\text{Ho}_{A∞}(\mathcal{B}, \mathcal{P})$ contains the "pro exp. loc. system"
"Then"\(\oplus\) Hodge, pers \((B, P) \rightarrow \text{Hom}(B, P)\) fully faithful

② In the H-free situation,
\[K^b \mathcal{P} \approx \text{Hodge, pers}(B, P) \approx \text{Hom}(B, P)\]

II. Application

\[
\begin{array}{ccc}
X_0 & \rightarrow & X \\
\downarrow & & \downarrow \\
0 & \rightarrow & C
\end{array}
\]

Assume \(\exists \, C^\infty C_1 X\), pnty sh on \(X_0\) or in the
H-free situation, \(H = H_{c, p, q}\)

Classical formula \(i^* j_* \exp \cdot \exp \cdot F\)

Unip. part: \(i^* j_* (F \otimes F^{-2\infty}) U?)\)

\(\oplus\) of rank p = unp.