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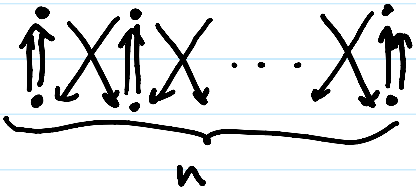
Cluster theory for Coherent Satake cat-y

Thm (C-Williams) The coherent Satake category $P^{GL_n}_{coh}(Gr_{GL_n})$ is a (rigid) monoidal catfn of the (quantum) cluster algebra of type



I- The ~~Affine~~ Grassmannian

$$G(k,n) = \{ 0 \subseteq V \subseteq \mathbb{C}^n \mid \dim V = k \}$$

Equivalently, fix $W = \mathbb{C}^n$,

$$Gr^{w_k}_{GL_n} = \{ \underset{\underset{W \otimes_{\mathbb{C}} \mathbb{C}[[t]]}{\|}}{L_0} \underset{k}{\subseteq} L \mid \begin{matrix} \dim(L/L_0) = k \\ \text{and } tL \subseteq L_0 \end{matrix} \}$$

$$= \{ L_0 \overset{k}{\subseteq} L \subseteq t^{-1} L_0 \} \simeq G(k,n)$$

Define a twisted product

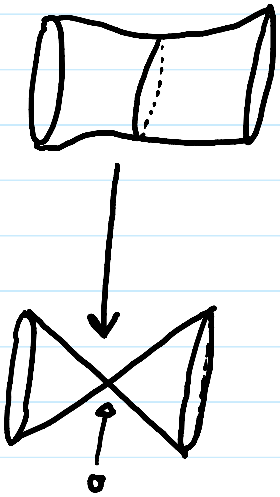
$$\begin{tikzcd}[column sep=small]
Gr^{\omega_{k_1}}_{GL_n} \tilde{\times} Gr^{\omega_{k_2}}_{GL_n} = \{ L_0 \arrow[d, "\text{forget } L_2"] & \overset{k_1}{\subseteq} & L_1 \arrow[ll, bend left, "t"] & \overset{k_2}{\subseteq} & L_2 \} \arrow[ll, bend left, "t"] \\
Gr^{\omega_{k_1}}_{GL_n}
\end{tikzcd}$$

makes this a $Gr^{w_{k_2}}_{GL_n}$ bundle over $Gr^{w_{k_1}}_{GL_n}$

Take $n=2$, $k=1$

Take $n=2$, $k=1$

$$\begin{tikzcd}
\mathbb{P}^1 \tilde{\times} \mathbb{P}^1 = \{ & L_0 & \overset{1}{\subset} & L_1 \arrow[ll, bend left] \arrow[d, "\pi = \text{forget } L_1"] & \overset{1}{\subset} & L_2 \arrow[ll, bend left] & \} \\
\{ & L_0 & & \overset{2}{\subset} & & L_2 \arrow[llll, bend left, "t^2"] & \} \quad =
\end{tikzcd}$$



Rmk $\pi_*\left(\mathbb{C}_{\mathbb{P}^1} \tilde{\boxtimes} \mathbb{C}_{\mathbb{P}^1}\right) = IC_X \oplus \mathbb{C}_0$

Moreover, taking cohomology, you get

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathrm{Sym}^2 \mathbb{C}^2 \oplus \mathbb{C}$$

2 (Constructible) Geometric Satake

G = s/s (red-ve) group, $K = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$,

$$Gr_G := G(K)/G(\mathcal{O})$$

$$G(K) \times_{G(\mathcal{O})} G(K)/G(\mathcal{O}) \xrightarrow{m} G(K)/G(\mathcal{O}),$$ analogously to

$$G \times_B G/B \xrightarrow{m} G/B$$

On Gr_G, we define

$P^{G(\mathcal{O})}(Gr_G)$ = abelian cat-y of perverse $G(\mathcal{O})$-eq. sheaves on Gr_G.

sheaves on Gr_G.

$P^{G(\mathcal{O})}(Gr_G)$ is a monoidal cat-y

Thm (geo. Satake) $P^{G(\mathcal{O})}(Gr_G) \underset{\text{monoidal}}{\simeq} Rep(G^\vee)$ ↙ Langlands dual

Rmk Kapustin-Witten geometric Langlands duality can be formulated as a 4d $\mathcal{N}=4$ SUSY gauge theory

3.- Coherent Satake Cat-y

$P^{G(\mathcal{O})}_{coh}$ = (abelian) cat-y of perverse coherent $G(\mathcal{O})$-eq. sheaves on Gr_G.

↑

monoidal, but not semisimple! ($\otimes$ is still exact)

(left & rt. duals exist, but do not coincide) not symmetric!

not braided! (still $\exists$ something like an R-matrix)

For GL_n, its structure is governed by a cluster algebra

4) Cluster algebras/Cat-s.

$$Q \rightsquigarrow \mathcal{A}_Q$$

e.g. $Q = \overset{\cdot 1}{\underset{2}{\Downarrow}}$ Original cluster x_1, x_2. $\mathcal{A}_Q \subseteq \mathbb{C}[x_1^{\pm1}, x_2^{\pm1}]$

Start with $\mathbb{C}[x_1, x_2] \subseteq \mathcal{A}_Q$

arrows $2 \to 1$ ↓ arrows into 1 ↙

Mutate at $\cdot 1$, new cluster $\{x_3, x_2\}$ $x_1 x_3 = x_2^2 + 1$

$\overset{3}{\Uparrow}$ ↗ ↑

[illegible]

new quiver $$\begin{tikzcd} 3 \\ 2 \arrow[u, shift left] \arrow[u, shift right] \end{tikzcd}$$

old new

And keep doing it...

$\mathcal{A}_Q$ = gen-d by all cluster variables obtained this way

$$\mathbb{C}[x_1, x_2, \frac{x_2^2+1}{x_1}, \dots]$$

In this case, we get $\mathbb{C}[x_i : i \in \mathbb{Z}]/(x_{i-1}x_{i+1} = x_i^2+1)$

(Monoidal) Cluster cat-y

Abelian, monoidal cat-y $\mathcal{C}_Q$.

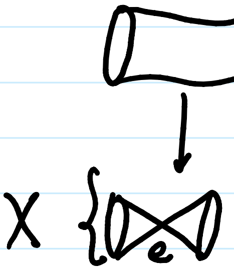
$$\left\{\text{Cluster variables } x_i\right\} \longrightarrow \left\{\text{Cluster objects } M_i\right\}$$

$$x_1 x_3 = x_2^2+1 \rightsquigarrow 0 \to M_2^2 \to M_1 M_3 \to Id \to 0$$

or

$$0 \to Id \to M_1 M_3 \to M_2^2 \to 0$$

Back to



We have

$$\begin{tikzcd} & \mathrm{Gr}^{\omega_1}_{GL_2} \arrow[dl] & \\ \mathcal{O}_{\mathbb{P}^1} \arrow[drr] & & \mathcal{O}_e \arrow[dll] \\ \mathcal{O}_{\mathbb{P}^1}(1) \arrow[u, shift left] \arrow[u, shift right] & & \mathcal{O}_e(1) \arrow[u, shift left] \arrow[u, shift right] \end{tikzcd}$$

Mutation?

$$0 \to \mathcal{O} \to \mathcal{O}(-1) \oplus \mathcal{O}(1) \to \overbrace{\mathcal{O} \oplus \mathcal{O}}^{\mathcal{O}_X} \to \mathcal{O}$$

$$0 \longrightarrow \mathcal{O}_e \longrightarrow \underbrace{\mathcal{O}_{\mathbb{P}^1}(-1) \star \mathcal{O}_{\mathbb{P}^1}(1)}_{\text{New object!}} \longrightarrow \overbrace{\mathcal{O}_{\mathbb{P}^1} \star \mathcal{O}_{\mathbb{P}^1}}^{e} \longrightarrow 0$$

<u>Rmk</u> Kapustin-Saulina introduced a holomorphic top. twist for 4d $\mathcal{N}=2$ gauge theory

line operators along $\mathbb{R}^2 \times pt \subseteq \mathbb{R}^2 \times \mathbb{C} \rightsquigarrow$ monoidal cat'y

Gaiotto-Moore-Neitzke: K(these cat's) are of the cluster algebras

<u>Costello</u>: For 4d $\mathcal{N}=2$ SYM w/ gauge group G, should be $\mathrm{Coh}^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$