Quantizations of Gieseke varieties & higher Calabi–Yau

1) $\mathbb{C}, n \in \mathbb{Z}_{>0}$

$R = \text{End}(G^n) \otimes \text{Hom}(G^n, G^n)$

$G = \text{GL}_n(G)$

$T^*R = R \otimes R^* = \text{End}(G^n) \otimes^2 \text{Hom}(G^n, G^n) \otimes \text{Hom}(G^n, G^n)$

$(A, B, i, j)$

$G \in T^*R$, w/momentum map

$\mu : T^*R \rightarrow g^* = g$

$\mu(A, B, i, j) = [A, B] - j^i$

$\mu^* : g \rightarrow \text{C}(T^*R)$

$\mu : \text{Vect}(R) \rightarrow \text{C}(T^*R)$

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- Hamiltonian reduction: $G \rightarrow G_\mu^*(0)$

$M(n, r) := \mu^{-1}(0)/G = \text{Spec}(\frac{\text{C}(T^*R)}{\text{C}(T^*R)/\mu^*(0)_G})$

Affine, singular, Poisson

Resoln. of singularities: GIT quotient $\theta = \det : G \rightarrow G^*$

$M^\theta(n, r) = \mu^{-1}(0)_{\theta-ss}$

you check $\text{Lie} A \times \mathbb{R}$ stable subs?
\[ M^{\theta}(n,r) = \text{smooth, irreducible symplectic variety of dim } = 2nr \]

\[ M^{\theta}(n,r) \to M(n,r) \text{ - resoln of singularities} \]

In what follows \( \text{End}(G) \to sl_n \)

\[ \implies \text{dim } = 2nr - 2 \]

Example: \( n = 1 \)

\[ M^{\theta}(1,r) = T^* \mathbb{P}^{r-1}, \text{ resolver } M(1,r) = \text{minimal nilp. orbit of } G \]

\[ r = 1 \]

\[ M(n,1) = \mathfrak{h} \otimes h^*/S, \quad h^* = \mathbb{C}^{n-1}, \text{ reductive fibres of } S, \]

\[ M^{\theta}(n,1) = \text{basically } \text{Hilb}_n(G^2) \]

**Quotization** \( \lambda \in G \)

\[ A_\lambda(n,r) = \left[ \frac{D(R)}{D(R) \otimes \mathbb{C}[\mathfrak{g}] - \text{Res}(\mathfrak{g})/\mathfrak{g} \otimes \mathfrak{g}} \right] G \]

An associ. algebra, w/ filtration by order of diff. op.

\[ \text{gr}_e A_\lambda(n,r) = \mathbb{C}[M(n,r)] = \mathbb{C}[M^{\theta}(n,r)] \]
Example \( n=1 \) \( A_2(1,1) = D^2(B^1) \)

\( r=1 \) \( A_2(n,1) = \text{spherical BCA of type } E_{ln} \)

Symmetry \( G^x \times GL(n) \cong R \)

\((x, h) (A, i) = (xA, hi)\)

Commuter w/ \( G \)-action

\( \Rightarrow C^x \times GL(n) \cong T^+E \cdot U(n, M^{\theta (n, n, x)}(\mathbb{C})) \)

\( d_2(n, r) \)

Hamiltonian actions

2) Finite dim. rep. of \( d_2(n, r) \)

Thm 1 1) \( \exists \) f.d. rep. of \( d_2(n, r) \) iff

\( \lambda = \frac{a}{n}, (a:n) = 1 \)

\( \lambda \in \mathbb{C} \setminus \{0\} \)

2) If so, \( d_2(n, r) \)-mod \( _{fd} \approx \text{ Vect} \)

Ex \( n=1 \) \( D^2(B^1) \leftarrow U(1) \)

\( \begin{array}{c}
\circ \downarrow \Gamma(O(n)) \lambda > 0 \\
U(1) \rightarrow \end{array} \)

\( r=1 \) \[ \text{[Berend-Etingof-Ginzburg]} \]

3) Construction of f.d. rep

\( D(R) \)-mod \( ^{G, \lambda} = \{ M \in D^2 \text{-mod with } G \cdot M \mid E_M = E_k \cdot \mathbb{H}(E)^{\lambda} \} \)
\[ D(R) - \text{mod}^{G, \alpha} = \left\{ \text{mod } D(R) \text{ with } G \text{CM} \mid \xi_m - \xi_k - h_k(\xi) \neq 0 \right\} \]

\[ \text{mod } D(R) - \text{mod}^{G, \alpha} \]

\[ R(M) = M^G \otimes D(R)^G \text{ factor via } A_2(n, r) \]

\[ \text{quotient factor } R : D(R) - \text{mod}^{G, \alpha} \longrightarrow A_2(n, r) - \text{mod} \]

**Fact**: \( \exists M^\alpha_{a, n} \in D(\mathfrak{gl}_n) - \text{mod}^{S_n} \) s.t.

\( a > 0, (a, n) = 1 \)

\( \cdot M^\alpha_{a, n} \text{ is irred} \)

\( \cdot \text{Supp}(M^\alpha_{a, n}) \leq \text{nilp core} \)

\( \cdot \exists (z_1, \ldots, z_r) \mid z^2 = 1^2 \text{ acts on } M^\alpha_{a, n} \text{ via the character } z \mapsto z^{-a}. \)

Intermediate ext from propd nilp orbit:

\[ \text{Ext}_{L^{\mathfrak{gl}_n}} G M^\alpha_{a, n} \text{ w/dual matrixer acting via } z \mapsto z^{-a} \]

\[ M^\alpha_{a, n} \in D(\mathfrak{gl}_n) - \text{mod}^{G, \alpha} \]

\[ \sim M^\alpha_{a, n} \otimes C[\text{Hom}(G', G')] \in D(R) - \text{mod}^{G, \alpha} \]

**Propn** (Etingof-Y.Keller-L)

\[ L_{a, n} := R(M^\alpha_{a, n} \otimes C[\text{Hom}(G', G')]) \]

is irred. f.d. \( A_2(n, r) \).

**Goal**: Compute dim \( L_{a, n} \).
Good: Compute \( \dim L_{a,n} \).

4) Higher rank Catalan \#s.

**Theorem (Calaque–Enriquez–Etingof '07)**

Mult. of \( G \)-rep \( V_n(\mu) \) (\( \mu = (\mu_0, \ldots, \mu_n) \))

\[ = \frac{1}{n} \dim V_n(\mu) \text{ in } M_n^G. \]

\[ \text{Hom}(C^\infty, C^\infty) = \bigoplus \nabla V_n(\nu) \otimes V_n(\nu)^* \]

\( \nu = \text{pot } w/ \leq \min(n, n - r, a) \)

\[ \Rightarrow L_{a,n} = \bigoplus \nabla V_n(\nu) \]

\( \nu, (\nu) = a \leq \min(n, n - r, a) \)

\[ \text{Claim} \]

\[ \dim L_{a,n} = \frac{1}{n} \left( \frac{(nr + a - 1)!}{a!} \right) \]

For \( r = 1 \): \( \frac{(n+a-1)!}{n!a!} \) a rational Catalan \#.

Adding stupid \( C^\infty \) turn \# into quotient \# r

**Questions**

1) Combinatorial meaning? Basis in \( L_{a,n} \), which is e-basis for central \( \mathfrak{g} \)-valued functions.
Coming from truncated shifted Young! 

Known for $r=1$

2) Where's the clear 1-diml tour?

$M(n,0,n) \mathcal{G}L \times \mathcal{G}^r \times \mathcal{G}^r$

only give a filtration

$\sim$ higher rank $(q,t)$-Catalan $\#s$?

Way #1: $Mg$ has Hodge filtration $\sim gr$...

Way #2: Use EHA $C \oplus K^r\text{Coh}_{\mathcal{G}L \times \mathcal{G}^r \times \mathcal{G}^r}(M(n,r))$

apply generator of slope $\delta$ to $1\delta$. 