§6. DARK crystals

Nov. 30

Overview

Tame nonsym. Catalan fun of partition weight $=$ char of affine generalized Demazure crystals

Schur/Key expansion of

Katabolism on tabloids

$\|$ match

subsets of tensor product of KR crystals (DARK crystal)

§6.1 Single row Kirrilov–Reshetikhin (KR) crystals

$B^{1,s}$ $\quad s \in \mathbb{Z}_{\geq 0}$ $\quad U_q'(\hat{sl}_\ell)$ – seminormal crystal

as a set: all weakly increasing words of length s in alphabet $[\ell]$

weight function $B^{1,s} \to P_\ell$

E.g. $b = 22 \in B^{1,2}$, $\ell = 3$

$i \in [\ell-1]$

$\tilde{e}_i(b)$: change its leftmost $i+1$ to i

$\tilde{f}_2(22) = 23$

$\tilde{f}_i(b)$: change its rightmost i to $i+1$

$\tilde{e}_0(b)$: remove a 1 from the beginning, add ℓ to the end (if no 1, $\tilde{e}_0(b) = 0$)

$\tilde{f}_0(b)$: remove an ℓ from the end, add 1 to begining (if no ℓ, $\tilde{f}_0(b) = 0$)

$\ell = 2$. $\quad \tilde{f}_0(22) = 12$

if $\ell = 3$ $\quad \tilde{f}_0(22) = 0$

$B^{1,0} = \{b_0\}$

$\downarrow$ empty word

§6.2 Products of KR crystals

$B^\mu = B^{1,\mu_p} \otimes B^{1,\mu_{p-1}} \otimes \cdots \otimes B^{1,\mu_1}$ $\quad \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p)$

elements in B^μ : biwords

$b = \begin{pmatrix} v_1 & \cdots & v_m \\ w_1 & & w_m \end{pmatrix}$ $\quad v_i, w_i \in \mathbb{Z}_{\geq 1}$ $\quad i < j \Rightarrow v_i \geq v_j$, $\quad v_i = v_j \Rightarrow w_i \leq w_j$

top(b) := $v_1 v_2 \cdots v_m$

bottom(b) := $w_1 \cdots w_m$

ith block of b

E.g. $\begin{pmatrix} \underbrace{\begin{matrix} 3333 \\ 2234 \end{matrix}}_{3^{rd}} & \underbrace{\begin{matrix} 22222 \\ 22333 \end{matrix}}_{2^{nd}} & \underbrace{\begin{matrix} 11111 \\ 11122 \end{matrix}}_{1^{st}} \end{pmatrix} \in B^{554}$

crystal operator on B^μ $\quad (i \in [\ell-1])$

E.g. $b = 2234 \ 22333 \ 11122$

$\qquad\ \ )\,)\,(\quad)\,)\,(\,(\,(\qquad)\,)$

$\tilde{e}_2(b) = 2234 \ 22233 \ 11122$ $\qquad$ leftmost unpaired $i+1 \to i$

$\tilde{f}_2(b) = 2234 \ 23333 \ 11122$ $\qquad$ rightmost unpaired $i \to i+1$

§6.3 RSK and crystal (review)

$b \in B^\mu \xrightarrow{\text{RSK}} (P(b), Q(b))$

insertion tableau $P(b)$

① row insertion, left to right on bottom(b)

② column insertion, right to left on bottom(b)

recording tableau (Q) : from ②

Thm (Shimozono) $B^\mu = \bigsqcup_{T \in SSYT_\ell(\mu)} C_T$ $\qquad C_T := \{ b \in B^\mu \mid Q(b) = T \} \cong B^{gl}(sh(T))$

$B^{gl}(\nu)$ = the highest weight $U_q(gl_e)$ - crystal of highest weight ν.

§6.4 The inv bijection and RSK

Tabloid $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Z}^{\ell}_{\geq 0}$ weak comp.

diagram of α : left adjusted, α_i boxes at row i

A tabloid T of shape α is a filling of diagram α w/ weakly increasing rows, in English notation.

shape$(T) = \alpha$ content$(T) = (c_1, \dots, c_p)$ $c_i = \#\, i$'s

$\mathrm{SSYT}_\ell(\mu) \subseteq \mathrm{Tabloids}_\ell(\mu) \subseteq \mathrm{Tabloids}_\ell$ μ = content (not shape!)

partition shape + strictly increase down columns

$\mathrm{Tabloids}_\ell = \bigsqcup_\mu \mathrm{Tabloids}_\ell(\mu)$

$B^\mu \xleftrightarrow{\text{inv}} \mathrm{Tabloids}_\ell(\mu)$

$b = \begin{pmatrix} 3333 & 22222 & 11111 \\ 2234 & 13334 & 11222 \end{pmatrix} \longmapsto \left(\begin{array}{c|cccc} 44 & 3333 & 22222 & 111 \\ 23 & 2223 & 11133 & 112 \end{array}\right)$

inv(b) =
1 1 2
1 1 1 3 3
2 2 2 3
2 3

P
1 1 2 2 2
4

Q
1 1 1 1 1
2

1 1 2 2 2
~~3 3~~ ~~3 3~~ 3 4 4 ← 3

1 1 1 1 1
2 2 2 2

1 1 1 2 2 2 ← 1
3 3 3 4 3 4
4

1 1 1 1 1 2 3
2 2 2 2 3 3
3
= Q(b)

1 1 1 2 2 2
3 3 3 3 3 4 ← 2
4

1 1 1 2 2 2 4
2 2 3 3 3 3
4
= P(b)

1 1 2
3 3

1 1 1 1 1 2
3 3 3

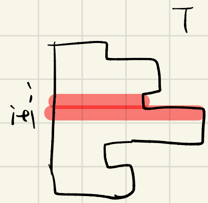
1 1 1 1 1 2 3
2 2 2 2 3 3
3
P(inv(b))

prop. $Q(b) = P(\mathrm{inv}(b))$, $\quad P(T) = Q(\mathrm{inv}(T))$

§6.5 Partial Insertion & $\tilde{e}_i^{max}$

$$\tilde{e}_i^{max} := \tilde{e}_i^{\varepsilon(b)}(b)$$

Partial Insertion $\quad P_i(T) \quad \underbrace{P(T^{i+1}T^{i})}$ $\rightarrow$



Eg. $P_2\left(\begin{array}{cccccc} 1&1&2&2&& \\ 1&2&3&3&4& \\ 1&1&2&2&2&3 \\ 2&3&4&&& \end{array}\right)$ $\quad P(112223\ 12334)$

$\uparrow\uparrow\uparrow\leftarrow$

12334   122334   1222334   12222334
23   23   23   23

$\boxed{\begin{array}{l} 1112222334 \\ 23 \end{array}}$

$$= \begin{array}{cccccccccc} 1&1&2&2&&&&&& \\ 1&1&1&2&2&2&2&3&3&4 \\ 2&3&&&&&&&& \\ 2&3&4&&&&&&& \end{array}$$

prop. 6.9 $\quad \forall\ b \in B^{\mu},\ i \in [\ell-1],\quad \tilde{e}_i^{max}(b) = \mathrm{inv}(P_i\, \mathrm{inv}(b))$

Eg.

$$\begin{pmatrix} 44 & 3333 & 2222222 & 11111 \\ 24 & 22\underline{34} & \underline{112}333 & \underline{34}\ 11233 \end{pmatrix} \overset{\mathrm{inv}}{\longleftrightarrow} \begin{pmatrix} 444 & 333333 & 22222 & 1111 \\ 234 & 112223 & 12334 & 1122 \end{pmatrix} \quad \begin{array}{cccccc} 1&1&2&2&& \\ 1&2&3&3&4& \\ 1&1&2&2&2&3 \\ 2&3&4&&& \end{array}$$

$\downarrow \tilde{e}_2^{max}$ $\downarrow P_2$

$$\begin{pmatrix} 44 & 3333 & 2222222 & 11111 \\ 24 & 2234 & 1122234 & 11222 \end{pmatrix} \overset{\mathrm{inv}}{\longleftrightarrow} \begin{pmatrix} 444 & 33 & 222222222 & 1111 \\ 234 & 23 & 111222334 & 1122 \end{pmatrix} \quad \begin{array}{ccccccccc} 1&1&2&2&&&&& \\ 1&1&1&2&2&2&3&3&4 \\ 2&3&&&&&&& \\ 2&3&4&&&&&& \end{array}$$

prop 6.10 $b \in B^\mu$, $T = \mathrm{inv}(b)$. Then b is a $U_q(\mathfrak{gl}_\ell)$ highest weight element iff any of the equivalent condition holds:

(a) $\tilde{e}_i(b) = 0 \quad \forall i \in [\ell-1]$

(b) $P_i(T) = T \quad \forall i \in [\ell-1]$

(c) T is a tableau. (semistandard)

prop 6.11 let $B^{\mathfrak{gl}}(\nu)$, $\nu = (\nu_1 \geq \cdots \geq \nu_\ell)$ be a highest weight $U_q(\mathfrak{gl}_\ell)$-crystal and u_ν its highest weight element. Then

(i) $\mathcal{F}_{w_0}\{u_\nu\} = B^{\mathfrak{gl}}(\nu)$

(ii) $\tilde{e}_1^{\max}, \tilde{e}_2^{\max} \cdots \tilde{e}_{\ell-1}^{\max}$ on $B^{\mathfrak{gl}}(\nu)$ satisfy the 0-Hecke relation

(iii) Operators $P_1, P_2, \cdots, P_{\ell-1}$ on Tabloids satisfy the 0-Hecke relation

(iv) $\tilde{e}_{w_0}^{\max}(b) = u_\nu$ for any $b \in B^{\mathfrak{gl}}(\nu)$

and $P_{w_0}(T) = P(T)$ for any $T \in \mathrm{Tabloids}_\ell$.

$$\tilde{e}_w^{\max} := \tilde{e}_{i_1}^{\max} \tilde{e}_{i_2}^{\max} \cdots \tilde{e}_{i_m}^{\max}, \qquad w = s_{i_1} s_{i_2} \cdots s_{i_m} \in H_\ell$$

P_w

$$\begin{tikzcd}
B^\mu \arrow[r, leftrightarrow, "\mathrm{inv}"] \arrow[d, "\tilde{e}_i^{\max}"'] & \mathrm{Tabloids}_\ell(\mu) \arrow[d, "P_i"] \\
B^\mu \arrow[r, leftrightarrow, "\mathrm{inv}"] & \mathrm{Tabloids}_\ell(\mu)
\end{tikzcd}
\qquad
\begin{tikzcd}
B^\mu \arrow[r, leftrightarrow, "\mathrm{inv}"] \arrow[d, "\tilde{e}_{w_0}^{\max}"'] \arrow[dr, "\theta"] & \mathrm{Tabloids}_\ell(\mu) \arrow[d, "P_{w_0}"] \\
B^\mu \arrow[r, leftrightarrow, "\mathrm{inv}"] & \mathrm{Tabloids}_\ell(\mu)
\end{tikzcd}$$

§6.6 The kat & kat' operators and the automorphism τ

Recall $\quad \sigma \in \Sigma = \{\tau^j \mid j \in [\ell-1]\}$

B, B' : $U_q'(\hat{sl}_\ell)$ - seminormal crystals

A **σ-twist** is a bijection $B \longrightarrow B'$ taking edge i to edge $\sigma(i)$.

w a word, $\quad \mathrm{sort}(w)$ = weakly increasing rearrangement

$m \in \mathbb{Z}$, $\quad \mathrm{mod}_1^\ell(m)$ = the unique $i \in [\ell]$ s.t. $i \equiv m \mod \ell$

prop/def There is a unique σ-twist $\mathcal{F}_\sigma : B^\mu \longrightarrow B^\mu \quad \forall\, \sigma \in \Sigma$.

$$\mathcal{F}_{\tau^{-1}}(\underset{\substack{\uparrow \\ B^{1,s}}}{v_1 \cdots v_s}) = \mathrm{sort}\big(\mathrm{mod}_1^\ell(v_1-1)\ \mathrm{mod}_1^\ell(v_2-1) \cdots \mathrm{mod}_1^\ell(v_s-1)\big)$$

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E.g. $\ell = 4 \qquad b = 233\ \underline{1124}\ 12223 \in B^{543}$ $\qquad$ "promotion"

$\mathcal{F}_{\tau^{-1}}(b) = 122\ 1344\ 11124 \in B^{543}$

Define $\quad \mathrm{kat}'(b) = \mathcal{F}_{\tau^{-1}}(b^p \cdots b^2) \in B^{(\mu_2, \ldots, \mu_p)}$.

E.g. $\quad \mathrm{kat}'(b) = 122\ 1344$

Katabolism $\quad T \in \mathrm{Tabloids}_\ell$, define $\mathrm{kat}(T) \in \mathrm{Tabloids}_\ell$

- remove all 1's from T, left adjust rows
- remove the first (top) row and add it as the new ℓ-th row
- subtract 1 from all letters

E.g.

$$T = \begin{array}{|llllll} \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} & 4 & 4 \\ 2 & 2 & 2 & 2 & 5 & \\ 3 & 3 & 3 & & & \\ 4 & 5 & 5 & & & \\ & & & & & \end{array} \quad \xrightarrow{\ \mathrm{kat}\ } \quad \begin{array}{|lllll} 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & & \\ 3 & 4 & 4 & & \\ & & & & \\ 3 & 3 & & & \end{array}$$

$\ell = 5$

empty row $\qquad$ empty row

Prop 6.13 $\forall\, T \in \mathrm{Tabloids}_\ell$, $\quad \mathrm{inv}(\mathrm{kat}(T)) = \mathrm{kat}'(\mathrm{inv}(T))$

E.g. ($\ell=4$),
$$b = \begin{pmatrix} 444 & 3333 & 22222 & 111111 \\ 233 & 1124 & 12223 & 111111 \end{pmatrix} \in B^{6543}$$

$$T = \mathrm{inv}(b) = \begin{pmatrix} 4 & 333 & 22222 & 111111111 \\ 3 & 244 & 22234 & 111111233 \end{pmatrix} = \begin{array}{|l} \underline{111111}233 \\ 22234 \\ 244 \\ 3 \end{array}$$

$$\mathrm{kat}(T) = \underbrace{\begin{array}{|l} 11123 \\ 133 \\ 2 \\ 122 \end{array}}_{T'} = \begin{pmatrix} 444 & 3 & 222 & 11111 \\ 122 & 2 & 133 & 11123 \end{pmatrix} \xrightarrow{\mathrm{inv}} \underbrace{\begin{pmatrix} 333 & 2222 & 11111 \\ 122 & 1344 & 11124 \end{pmatrix}}_{b''}$$

$$\mathrm{kat}'(b'') = 122 \quad 1344 \quad 11124$$

$$\begin{tikzcd} b \arrow[r, "\mathrm{inv}"] \arrow[d, dashed, "\mathrm{kat}'"'] & T \arrow[d, "\mathrm{kat}"] \\ b' & T' \arrow[l, "\mathrm{inv}"'] \end{tikzcd}$$

§6.7 Katabolism

The Kirillov-Reshetikhin affine Demazure (DARK) crystal associated to $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$ and $\underline{w} = (w_1, \ldots, w_p) \in (H_\ell)^p$ is the subset of B^μ

$$B^{\mu;\underline{w}} := \mathcal{F}_{w_1}(\mathcal{F}_\tau \mathcal{F}_{w_2}(\cdots \mathcal{F}_\tau \mathcal{F}_{w_{p-1}}(\mathcal{F}_\tau \mathcal{F}_{w_p}\{b_{\mu_p}\} \otimes b_{\mu_{p-1}}) \cdots \otimes b_{\mu_2}) \otimes b_{\mu_1})$$

$\mathcal{F}_{w_i} := \mathcal{F}_{j_1} \cdots \mathcal{F}_{j_k}$ for any $w_i = s_{j_1} \cdots s_{j_k}$

$\mathcal{F}_\tau$: add 1 (mod ℓ) to every letter, then sort $\nearrow$

Thm 6.20 For $\mu = (\mu_1 \geq \cdots \geq \mu_\ell \geq 0)$ and $\underline{w} = (w_1, \cdots w_p) \in (H_\ell)^p$

$$B^{\mu;\underline{w}} \overset{\mathrm{inv}}{\longleftrightarrow} \{T \in \mathrm{Tabloids}_\ell(\mu) \mid T \text{ is } \underline{w}\text{-katabolizable}\}$$

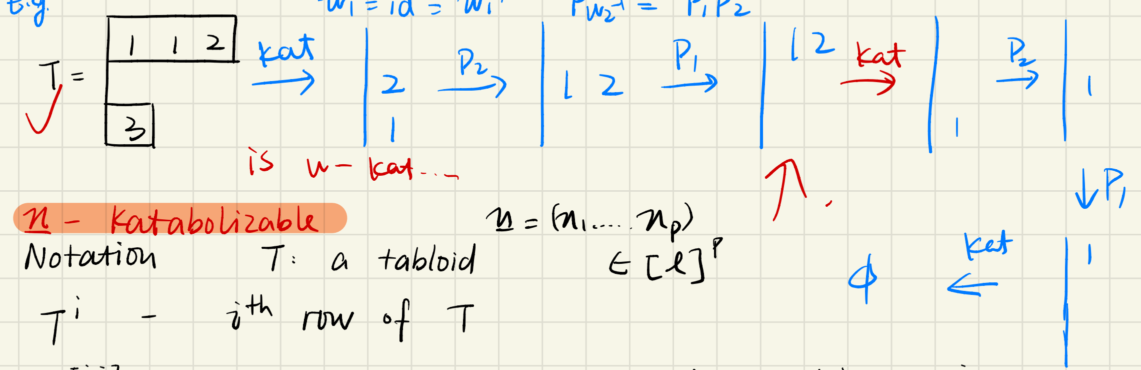
w - katabolizable

Def'n Let $\underline{w} = (w_1, \dots, w_p) \in (\mathcal{H}_\ell)^p$. $T \in \text{Tabloids}_\ell$ is $\underline{w}$ - katabolizable if all the 1's of $P_{w_1^{-1}}(T)$ lie in its first row and $\mathrm{kat}(P_{w_1^{-1}}(T))$ is $(w_2, \dots, w_p)$ - katabolizable.
If $\underline{w}$ is the empty sequence, the only $\underline{w}$ - katabolizable tabloid is the empty one.

$P_{w_1^{-1}} = P_{i_1} P_{i_2} \cdots P_{i_k}$ $\quad w_1^{-1} = s_{i_1} \cdots s_{i_k}$.

$\rightarrow$ $w = (id, s_2 s_1, s_2 s_1)$ $\quad \ell = 3$ $\quad P_{w_3^{-1}} = P_1 P_2$

E.g. $w_1 = id = w_1^{-1}$ $\quad P_{w_2^{-1}} = P_1 P_2$



$\underline{n}$ - katabolizable

$\underline{n} = (n_1, \dots, n_p)$

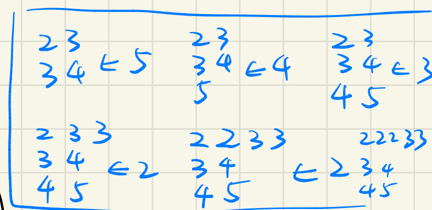
Notation $\quad T$: a tabloid $\quad \in [\ell]^p$

T^i - i^{th} row of T

$T^{[i,j]}$ - subtabloid of T consisting of row $i, i+1, \dots, j$

$T \in \text{Tabloids}_\ell$ s.t. $T^{[i,\ell-1]}$ is a tableau, define $P_{i,\ell}(T)$

- Fix row $1 \sim i-1$
- column insert T^ℓ into $T^{[i,\ell-1]}$



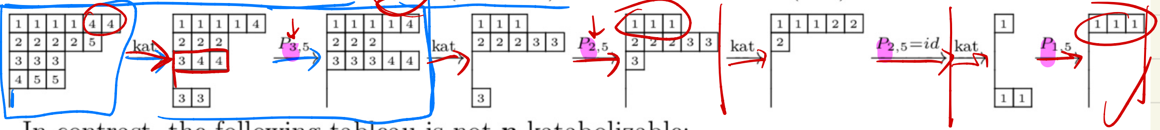
E.g. $\ell = 5$.

$$P_{2,5}\begin{pmatrix} 1 & 1 & 1 & 1 & 4 & 5 \\ \boxed{\begin{matrix} 2 & 3 \\ 3 & 4 \end{matrix}} \\ \\ 2 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 4 & 5 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}$$

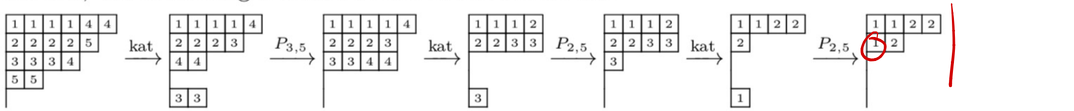
column

Let $\underline{n} = (n_1, \dots, n_{p-1}) \in [\ell]^{p-1}$ and $\mu \in \mathbb{Z}^p_{\geq 0}$. A tableau $T \in SSYT_\ell(\mu)$ is $\underline{n}$ - katabolizable if, for all $i \in [p-1]$, the tabloid $P_{n_i,\ell} \circ \mathrm{kat} \circ \cdots \circ P_{n_2,\ell} \circ \mathrm{kat} \circ P_{n_1,\ell} \circ \mathrm{kat}(T)$ has all its 1's on the first row.

Example 2.16. For $\ell = 5$ and $\mathbf{n} = (3, 2, 2, 1)$, the tableau below (left) is $\mathbf{n}$-katabolizable:



In contrast, the following tableau is not $\mathbf{n}$-katabolizable:



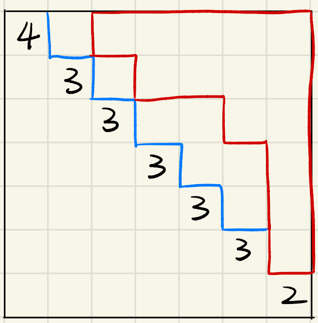
In some cases, can just use $\underline{n}$-katabolizable (easier) instead of $\underline{w}$-katabolizable.

prop 6.15 Let $\mu \in \mathbb{Z}^p_{\geq 0}$ and $\underline{n} = (n_1, \cdots, n_{p-1}) \in [\ell]^{p-1}$ satisfies $n_{i+1} \geq n_i - 1$ $\forall\, i \in [p-2]$. Then

$U \in SSYT_\ell(\mu)$ is $\underline{n}$-katabolizable $\iff$ $(id, s(n_1), \cdots, s(n_{p-1}))$-katabolizable

<u>Notation</u> $s(d) := s_{\ell-1} \cdots s_d \in H_\ell$ $\quad d \in [\ell]$

E.g. 6.18 $\ell = 7$, $\mu = 4333332$ $\quad \Psi$ root ideal $\quad \underline{n} = \underline{n}(\Psi)$



Recall $n_i \coloneqq \underline{n}(\Psi)_i = \#\{ j \in \{i, i+1, \cdots, \ell\} : (i,j) \notin \Psi \}$

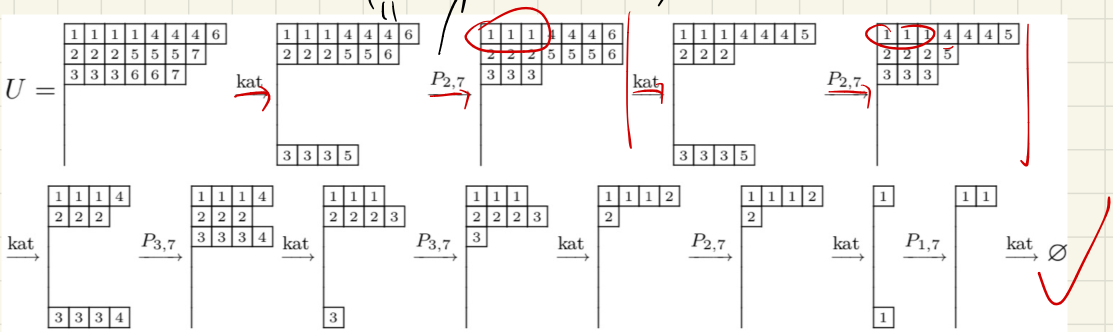
$\underline{n} = (2, 2, 3, 3, 2, 1)$ satisfies 6.15

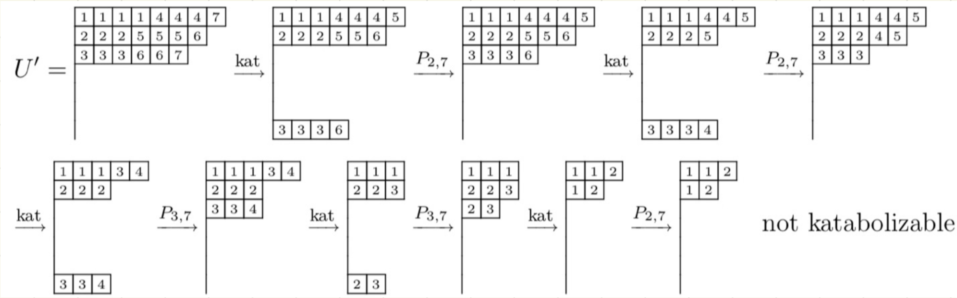
$\underline{w} = (id, \underline{s}(\Psi)) = (id, s_6 s_5 s_4 s_3 s_2, —, s_6 s_5 s_4 s_3, —$

$s(\Psi) = (s(n_1), s(n_2), \cdots)$.

$P_2P_3P_4P_5P_6(T)$

T_{11}





Thm 6.20 For $\mu = (\mu_1 \geq \cdots \geq \mu_\ell \geq 0)$ and $\underline{w} = (w_1, \cdots w_p) \in (\mathfrak{S}_\ell)^p$

$$B^{\mu;\underline{w}} \xleftrightarrow{\text{inv}} \{T \in \text{Tabloids}_\ell(\mu) \mid T \text{ is } \underline{w}\text{-katabolizable}\}$$

$+$

Thm (Shimozono) $B^{\mu} = \bigsqcup_{T \in SSYT_\ell(\mu)} C_T \qquad C_T := \{b \in B^{\mu} \mid Q(b) = T\} \cong B^{gl}(sh(T))$

$\Downarrow$

Thm 6.21 For $w_1 = w_0$, the DARK crystal $B^{\mu;\underline{w}}$ (regarded as a subset of the $U_q(gl_\ell)$-crystal B^{μ}) is a disjoint union of highest weight $U_q(gl_\ell)$-crystals, w/ decomposition given by

$$B^{\mu;\underline{w}} = \bigsqcup_{\substack{U \in SSYT_\ell(\mu) \\ U \text{ is } (id, w_2, \cdots, w_p)\text{-katabolizable}}} C_U, \qquad C_U = \{b \in B^{\mu} \mid Q(b) = U\}$$

Thm 2.17 μ a partion, root ideal Ψ, inv gives a bijection

$$B^{\mu; (w_0, \underline{s}(\Psi))} \xrightarrow{\text{inv}} \{T \in \text{Tabloids}_\ell(\mu) \mid P(T) \text{ is } \underline{n}(\Psi)\text{-katabolizable}\}$$

which takes content to shape. $P(T)$ = insertion tableau of $T^\ell \cdots T^1$ (row reading)

☆ **Thm 2.18** Any root ideal $\Psi \subset \Delta_\ell^+$ and partition $\mu = (\mu_1 \geq \cdots \geq \mu_\ell \geq 0)$,

$$H(\Psi; \mu; w_0)(X; q) = \sum_{\substack{U \in SSYT_\ell(\mu) \\ U \text{ is } \underline{n}(\Psi)\text{-katabolizable}}} q^{\text{charge}(U)} s_{\text{shape}(U)}$$