§4.1 $U_q(g)$ - seminormal crystals

g - sym. Kac-Moody Lie alg.; $U_q(g)$.

I - nodes for Dynkin diagram.

P^* - coweight lattice $\quad \{\alpha_i^\vee\} \subseteq P^*$

$P = \mathrm{Hom}_{\mathbb{Z}}(P^*, \mathbb{Z})$ weight lattice $\quad \{\alpha_i\}_{i\in I} \subseteq P$.

<u>Def:</u> A $U_q(g)$ - seminormal crystal is a set B:

- $wt : B \longrightarrow P$
- crystal ops: $f_i, e_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}$ $\quad f_i(0) = e_i(0) = 0$

$\ni$ 1) $wt(e_i b) = wt(b) + \alpha_i$
$wt(f_i b) = wt(b) - \alpha_i$

2) $\varepsilon_i(b) = \max\{k \geq 0 \mid e_i^k(b) \neq 0\} < \infty$

$\phi_i(b) = \max\{k \geq 0 \mid f_i^k(b) \neq 0\} < \infty$

$$\underbrace{e_i^k(b) \to \cdots \to b}_{\varepsilon_i(b)} \underbrace{\to \cdots \to f_i^k(b)}_{\phi_i(b)}$$

$\langle \alpha_i^\vee, wt(b) \rangle = \phi_i(b) - \varepsilon_i(b)$

<u>Def:</u> A strict embedding of $B \longrightarrow B'$
is an injective map: $\Psi : B \cup \{0\} \longrightarrow B' \cup \{0\}$

$\Psi(0) = 0$ & Ψ commutes w/ $wt, f_i, e_i, \varepsilon_i, \phi_i \ \forall i$

<u>Note:</u> This embedding maps B to a disjoint union of connected components of B'

<u>Def:</u> The tensor prod of $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$

w/ $wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$.

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i(b_1) \otimes b_2 & \varepsilon_i(b_1) > \phi_i(b_2) \\ b_1 \otimes e_i(b_2) & \varepsilon_i(b_1) \leq \phi_i(b_2) \end{cases}$$

$$f_i(b_1\otimes b_2) = \begin{cases} f_i(b_1)\otimes b_2 & \varepsilon_i(b_1)\geq \phi_i(b_2)\\ b_1\otimes f_i(b_2) & \varepsilon_i(b_1)<\phi_i(b_2)\end{cases}$$

Def: $\mathcal{O}_{int}$ = cat of $U_q(g)$-mods $M \ni M \simeq \bigoplus M_\lambda$

$\uparrow$ int. h.w. $U_q(g)$-mod.

(if we assume roots & coroots are lin indep).

Then for any $M \in \mathcal{O}_{int}$ has a <u>unique</u> crystal basis $(\mathcal{L}, B)$

$\Rightarrow$ yields a $U_q(g)$ - seminormal crystal.

Def: the $U_q(g)$-crystal is the seminormal crystal that arises in this way

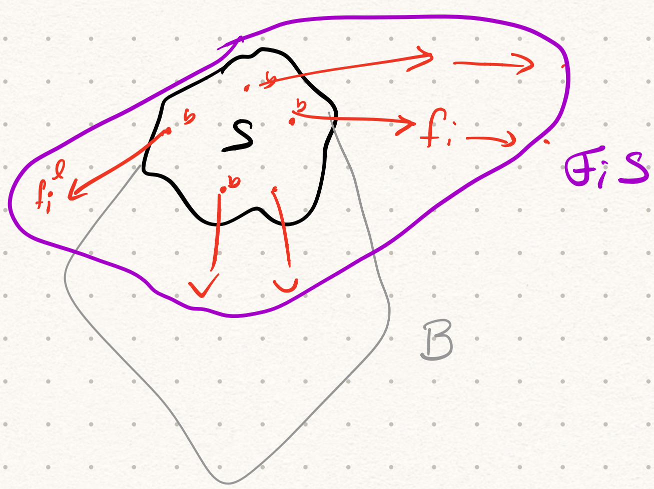
Def: The highest weight $U_q(g)$-crystal $B(\Lambda)$ is the crystal for the irred. h.w. module $V(\Lambda) \in \mathcal{O}_{int}$; $\Lambda \in P^+$

$\Rightarrow$ Any $U_q(g)$ crystal is a disjoint union of h.w. $U_q(g)$-crystals.

§.4.2

For $S \subseteq B$; $i \in I$ let

$$\mathcal{F}_i S := \{f_i^m b \mid b \in S;\ m \geq 0\} \setminus \{0\} \subseteq B$$

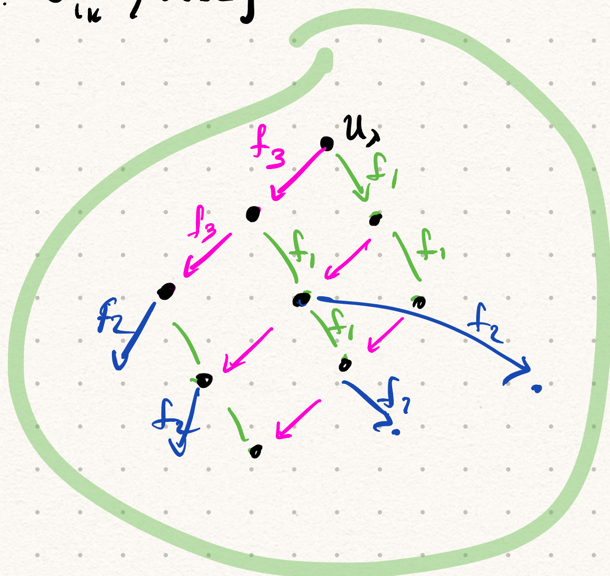


let u_Λ the highest weight element of $B(\Lambda)$

Definition: A $U_q(g)$ - Demazure crystal is the subset of a h.w. $U_q(g)$ crystal $B(\Lambda)$ of the form.

$$\tilde{F}_{i_1} \cdots\cdots \tilde{F}_{i_k}\{u_\Lambda\}$$

Ex: $\tilde{F}_2 \tilde{F}_1 \tilde{F}_3 \{u_\lambda\}$



let $J \subseteq I$ $\quad \hat{P}^* \subseteq P^*$ $\ni$ $\{\alpha_i^\vee\}_{i\in J} \subseteq \hat{P}^*$

$\exists$ proj $z: P \longrightarrow \hat{P} := \mathrm{Hom}_{\mathbb{Z}}(\hat{P}^*, \mathbb{Z})$

Suppose $\{\alpha_i^\vee\}_{i\in I}$, $\{\alpha_i\}_{i\in I}$ and $\{z(\alpha_i)\}_{i\in J}$ are lin independent

Def: $U_q(g_J) \subseteq U_q(g)$ sub. alg. gen. $\{e_i, f_i\}_{i\in J}$ and $\{q^h\}_{h\in\hat{P}^*}$

actually a q. env. alg. for the cartan datum above.

Since $(\mathcal{L}, B)$ is a crystal basis for M as $U_q(g)$-mod and as a $U_q(g_J)$-mod. then:

Def: $\hat{B}$ = the associated $U_q(g_J)$-crystal of $\mathrm{Res}_{U_q(g_J)}(\mathcal{L}, B)$

$\hat{B}$ is obtained by replacing $\mathrm{wt} \longmapsto z\circ \mathrm{wt}$

and consider only $\{f_i, e_i\}_{i\in J}$

Denote $\vec{B} = \mathrm{Res}_J B$

Theorem: (4.1) For any $U_q(g)$ Demazure crystal S then its restriction $\mathrm{Res}_J S$ is isomorphic to a disjoint union of $U_q(g_J)$ - Demazure crystals.

Proof:

$$\begin{array}{ccccl} S = \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_k}\{u_\lambda\} & \subseteq & B(\Lambda) & \leftarrow & \text{h.w. } U_q(g)\text{-crystal.} \\ \downarrow & & \downarrow & & \\ \mathrm{Res}_J(S) & \subseteq & \mathrm{Res}_J(B(\Lambda)) & \simeq & \bigsqcup \text{ highest weight } U_q(g_J)\text{-crystals.} \end{array}$$

Since $\{\alpha_i^\vee\}_{i\in I}$ are lin independent then we can choose

$$\{\Lambda_j\}_{j\in I} \subseteq P \ni \langle \alpha_i^\vee, \Lambda_j\rangle = m\,\delta_{ij} \quad i,j\in I$$
$$m \in \mathbb{Z}_{\geq 1}$$

Set $\bar{J} = I \setminus J$ $\qquad P_{\bar{J}} := \sum_{i\in \bar{J}} \Lambda_i$

$$c := 1 + \max_i \{\varepsilon_i(b) \mid b \in S\}$$

Consider $B(cP_{\bar{J}})$ - a $U_q(g)$-crystal w/ h.w. $u_{cP_{\bar{J}}}$

$\forall b \in S.$ $\phi_i(u_{cP_{\bar{J}}}) > \varepsilon_i(b)$

since $\langle \alpha_i^\vee, wt(b)\rangle = \phi_i(b) - \varepsilon_i(b)$

$\phi_i(u_{cP_{\bar{J}}}) = \langle \alpha_i^\vee, cP_{\bar{J}}\rangle = c\cdot m \qquad i \in \bar{J}$

so $f_i(b\otimes u_{cP_{\bar{J}}}) = b \otimes f_i(u_{cP_{\bar{J}}}) \notin S \otimes \{u_{cP_{\bar{J}}}\}$

Recall thm (Joseph) $\forall \lambda \in P^+$; $\mu \in P^+ \cup \{\infty\}$

$\underbrace{B_w(\mu)}_{\text{Dem. crystal w/ h.w. } u_\mu} \otimes \{u_\lambda\}$ is a disjoint union of Demazure crystals

Note: $\uparrow$ in the original paper the $\otimes$-product rule is Backwards
$\Rightarrow$ the thm. holds for $u_\lambda \otimes B_w(\mu)$.

$\Rightarrow$ $S \otimes \{u_{cp_J}\} = \bigsqcup \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{ b_r \otimes u_{cp_J} \}$ for some $b_r \in S$.
and $i_j \in J$

$\Downarrow$
$e_i(b_r \otimes u_{cp_J}) = 0$

Now, $\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{ b_r \otimes u_{cp_J} \} = (\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{b_r\}) \otimes u_{cp_J}$ $\Downarrow$
$(b/c \ \phi_i(u_{cp_J}) > \varepsilon_i(b) \ \forall b \in S)$ $e_i(b_r) = 0$

and $\phi_i(u_{cp_J}) = \langle \alpha_i^\vee, cp_J \rangle = 0 \quad \forall i \in J$

So then $S = \bigsqcup \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{ b_r \}$

where $\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{b_r\}$ are $U_q(g_J)$-Dem. crystals. $\blacksquare$
$e_i(b_r) = 0.\ \forall i \in J.$