Rotation Theorem

Notations: $[d] = \{1, 2, \dots, d\}$

$\underline{x} = (x_1, x_2, \dots, x_\ell)$

$\underline{x}^{\pm 1} = (x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_\ell^{\pm 1})$

$\alpha \in \mathbb{Z}^\ell$, $\underline{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_\ell^{\alpha_\ell}$

$[\![\cdot]\!]$ – coefficient of $\cdot$ in monomial/key polynomial expansion

Plan
- (Tame) nonsymmetric Catalan functions
- Polynomial truncation operator
- Relations with Demazure operators, π_i, Φ
- Proof Rotation Theorem (Theorem 2.3)

Preliminaries

Symmetric group, S_ℓ

Generators: $s_1, s_2, \dots, s_{\ell-1}$

Relations:

$s_i^2 = 1 \quad \forall i \in [\ell-1]$

$s_i s_j = s_j s_i \quad \forall i, j \in [\ell-1]$ s.t. $|i-j|>1$

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \forall i \in [\ell-2]$.

0-Hecke monoid, $\mathcal{H}_\ell$

Generators: $\sigma_1, \sigma_2, \dots, \sigma_{\ell-1}$

Relations:

$\sigma_i^2 = \sigma_i$

$\sigma_i \sigma_j = \sigma_j \sigma_i$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Affine symmetric group, $\widehat{S}_\ell$

Generators: $s_1, s_2, \dots, s_{\ell-1}, \tau$

Additional relations : $\tau^{l} = 1$

$\tau s_i \tau^{-1} = s_{i+1} \quad \forall i \in [l-2]$

Affine 0-Hecke monoid $\hat{\mathcal{H}}_l$ defined similarly

In $\mathcal{H}_l$ $\quad w = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_m}$ is <u>reduced</u> if m is the smallest among all equivalent expressions in $\mathcal{H}_l$

$\text{length}(w) = m$

$\hat{\mathcal{H}}_l \quad w = \tau^k \underline{w'}$ ← can be expressed in σ_i's

$\text{length}(w) = \text{length}(w')$

S_l acts on $\mathbb{Z}^l$ $\quad s_i(\alpha) = s_i(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_l)$

$= (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_l)$

acts on $\mathbb{Z}[\underline{x}]$ $\quad s_i(\underline{x}^{\alpha}) = \underline{x}^{s_i \alpha}$.

$\mathcal{H}_l$ acts on $\mathbb{Z}^l$ $\quad \sigma_i(\alpha) = \begin{cases} s_i \alpha, & \text{if } \alpha_i > \alpha_{i+1} \\ \alpha, & \text{else} \end{cases}$

acts on $\mathbb{Z}[\underline{x}]$: $\sigma_i(\underline{x}^{\alpha}) = \underline{x}^{\sigma_i \alpha}$

<u>Demazure operators</u> $\quad \pi_i(f) = \dfrac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}$

$\pi_w(f) = \pi_{i_1} \cdots \pi_{i_m}(f)$ ← well-defined as π_i's satisfy similar relations in 0-Hecke monoid.

$w = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_m} \in \mathcal{H}_l$

<u>Key polynomials</u>, $\alpha \in \mathbb{Z}^l$

$K_\alpha := \pi_{p(\alpha)} \underline{x}^{\text{sort}(\alpha)}$

$\text{sort}(\alpha)$ — α rearranged in weakly decr. order

sort(α) — α rearranged in weakly decr. order
$p(\alpha)$ — elt. in $\mathcal{H}_\ell$ of shortest length
$\alpha \longrightarrow$ sort(α)

Notes: α - weakly decreasing $\Rightarrow K_\alpha = \underline{x}^\alpha$
α - weakly increasing $\Rightarrow K_\alpha = S_{sort(\alpha)}$

1. $\pi_i K_\alpha = K_{\sigma_i \alpha}$
2. $s_i(f) = f \Leftrightarrow \pi_i(f) = f \Leftrightarrow f$ is symm in x_i, x_{i+1}
3. $(x_1 x_2 \dots x_\ell)^d K_\alpha = K_{\alpha + (d,d,\dots,d)}$
4. [Reiner-Shimozono, 1995]:
$\{K_\alpha \mid \alpha \in \mathbb{Z}^\ell_{\geq 0}\}$ is a $\mathbb{Z}[q]$-basis for $\mathbb{Z}[q][\underline{x}]$

By 3 $\Rightarrow \{K_\alpha \mid \alpha \in \mathbb{Z}^\ell\}$ is a $\mathbb{Z}[q]$-basis for $\mathbb{Z}[q][\underline{x}^{\pm 1}]$

Polynomial truncation operator, poly on $\mathbb{Z}[q][\underline{x}^{\pm 1}]$

Def: $\text{poly}(K_\alpha) = \begin{cases} K_\alpha, & \text{if } \alpha \in \mathbb{Z}^\ell_{\geq 0} \\ 0, & \text{else} \end{cases}$

Extend poly on $\mathbb{Z}[[q]][\underline{x}^{\pm 1}]$ by $\text{poly}\left(\sum_{d \geq 0} f_d q^d\right)$
$= \sum_{d \geq 0} \text{poly}(f_d)\, q^d$

$\Psi \subseteq \Delta^+_\ell$ root ideal

Def: A labeled root ideal of length ℓ is a triple $(\Psi; \gamma; w)$
where $\Psi \subseteq \Delta^+_\ell$ root ideal
$\gamma \in \mathbb{Z}^\ell$
$w \in \mathcal{H}_\ell$

The nonsymmetric Catalan function for labeled root ideal (Ψ, γ, w)

The <u>nonsymmetric Catalan function</u> for labeled root ideal (Ψ, γ, w) of length ℓ is

$$H(\Psi; \gamma; w)(\underline{x}; q) = \pi_w \circ \mathrm{poly}\Big(\prod_{(i,j)\in\Psi} (1 - q\, x_i/x_j)^{-1} \underline{x}^{\gamma} \Big)$$

$$\mathbb{Z}[q][\underline{x}]$$

In the case $w = w_0 = \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_2\sigma_1)$
"longest elt in $\mathcal{H}_\ell$"

$\pi_{w_0} \circ \mathrm{poly}$ "=" π described in Eugene's talk

$$\pi : \mathbb{Z}[q][\underline{x}^{\pm 1}] \to \mathbb{Z}[q][\underline{x}^{\pm 1}]$$
$$\underline{x}^{\gamma} \mapsto s_\gamma$$

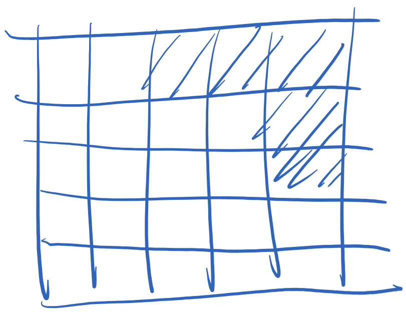
$$\Rightarrow H(\Psi; \gamma; w_0)(\underline{x}, q) = H(\Psi; \gamma)(\underline{x}; q)$$

<u>Statement of main theorem</u>

<u>Def</u>: $\Psi \subseteq \Delta^+_\ell$ root ideal
For each $i \in [\ell-1]$, $n_i = \big| \{ j \in \{i, i+1, \dots, \ell\} \mid (i,j) \notin \Psi \} \big|$
= # boxes on or above diagonal not in Ψ in row i.

$$n(\Psi) = (n_1, n_2, \dots, n_{\ell-1})$$

<u>Eg.</u> Ψ:



$n(\Psi) = (2, 3, 2, 2)$

<u>Def</u>: A labeled root ideal (Ψ, γ, w) of length ℓ is <u>tame</u> if $\{ n(\Psi)_1 + 1, \dots, \ell-2, \ell-1 \} \subseteq \{ i \in [\ell-1] \mid w\sigma_i = w \}$

non-descent of w

Right descent of w

If $(\Psi;\gamma,w)$ is tame, then $H(\Psi;\gamma;w)$ is tame.

Why tame $H(\Psi;\gamma;w)$?

- characters of gen. Demazure $U_q(\widehat{\mathfrak{sl}}_\ell)$ -crystals* (Thm 2.8)
- key positive, positive comb. formulas for coefficients using DARK crystals (Cor 7.13)
- $t=0$ nonsymmetric Macdonald polynomials $E_\alpha(\underline{x};q;0)$ are nonsymmetric Catalan functions (Thm 8.11)

Consider $\mathbb{Z}[q]$-alg. homom.

$$\Phi : \mathbb{Z}[q][\underline{x}] \longrightarrow \mathbb{Z}[q][\underline{x}]$$
$$x_i \mapsto x_{i+1} \quad \forall i \in [\ell-2]$$
$$x_\ell \mapsto q x_1$$

(Main Thm) Let $(\Psi;\gamma;w)$ be tame with $\gamma \in \mathbb{Z}^\ell_{\geq 0}$

Write $n(\Psi) = (n_1, n_2, \dots, n_{\ell-1})$

$\sigma(d) := \sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_d \in \mathfrak{S}_\ell \quad \forall d \in [\ell]$

$$\Rightarrow H(\Psi;\gamma;w) = \pi_w\, x_1^{\gamma_1}\left(\Phi\, \pi_{\sigma(n_1)} x_1^{\gamma_2}\right)\left(\Phi\, \pi_{\sigma(n_2)} x_1^{\gamma_3}\right)\cdots\left(\Phi\, \pi_{\sigma(n_{\ell-1})} x_1^{\gamma_\ell}\right)$$

Prop: [Recursion for $H(\Psi;\gamma;w)$] — sim. to Monica's talk.

Let $(\Psi;\gamma,w)$ — labeled root ideal of length ℓ

Suppose $\alpha \in \Psi$ is a removable root, i.e. $\Psi\setminus\alpha$ is again a root ideal.

Write $\epsilon_\alpha = \epsilon_a - \epsilon_b$ if $\alpha = (a,b)$

Then $H(\Psi;\gamma;w) = H(\Psi\backslash\alpha,\gamma;w) + q H(\Psi;\gamma+\epsilon_\alpha;w)$

<u>Sketch:</u> Apply $\pi_w \circ$ poly to:

$$\prod_{(i,j)\in\Psi}\left(1-q\frac{x_i}{x_j}\right)^{-1}\underline{x}^{\gamma} = \prod_{(i,j)\in\Psi\backslash\alpha}\left(1-q\frac{x_i}{x_j}\right)^{-1}\underline{x}^{\gamma} + q\prod_{(i,j)\in\Psi}\left(1-q\frac{x_i}{x_j}\right)^{-1}\underline{x}^{\gamma+\epsilon_\alpha}$$

<u>Polynomial truncation on monomials</u>

<u>Def:</u> [Fu, Lascoux, 2009] $f, g \in \mathbb{Z}[q][\underline{x}^{\pm1}]$

Consider bilinear form

$$(f,g) := [[\underline{x}^0]]\left(f(x_1,x_2,\dots,x_\ell)\,g(x_\ell^{-1},x_{\ell-1}^{-1},\dots,x_1^{-1})\prod_{(i,j)\in\Delta_\ell^+}\left(1-\frac{x_i}{x_j}\right)\right)$$

Key polynomials: $K_\alpha = \pi_{p(\alpha)}\,\underline{x}^{\mathrm{sort}(\alpha)}$

Demazure atoms: $\hat{K}_\beta = \hat{\pi}_{p(\beta)}\,\underline{x}^{\mathrm{sort}(\beta)}$

where $\hat{\pi}_i = \pi_i - 1$, $\hat{\pi}_w = \hat{\pi}_{i_1}\hat{\pi}_{i_2}\cdots\hat{\pi}_{i_m}$

if $w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}$.

[Fu, Lascoux, 2009]: $\{K_\alpha \mid \alpha\in\mathbb{Z}^\ell_{\geq 0}\}$ and $\{\hat{K}_\beta \mid \beta\in\mathbb{Z}^\ell_{\geq0}\}$ are dual bases

in $\mathbb{Z}[q][\underline{x}]$ wrt. $(\cdot,\cdot)$, $(K_\alpha, \hat{K}_{w_0\beta}) = \delta_{\alpha,\beta}$

[Blasiak, Morse, Pun, 2020] $\{K_\alpha \mid \alpha\in\mathbb{Z}^\ell\}$ and $\{\hat{K}_\beta \mid \beta\in\mathbb{Z}^\ell\}$ are dual bases

in $\mathbb{Z}[q][\underline{x}^{\pm1}]$ in the same sense.

Let $\hat{K}_\alpha = \sum_{\beta \in \mathbb{Z}^\ell} c_{\alpha,\beta} \, \underline{x}^\beta$

$$[[K_\alpha]] f = \left(f, \hat{K}_{w_0 \alpha}\right)$$

$$= [[\underline{x}^0]] \left(f \, \hat{K}_{w_0\alpha}(x_\ell^{-1}, \dots, x_1^{-1}) \prod_{(i,j)\in\Delta_\ell^+} \left(1 - x_i/x_j\right) \right)$$

$$= [[x^0]] \left(f \prod_{(i,j)\in\Delta_\ell^+} \left(1 - x_i/x_j\right) \sum_{\beta\in\mathbb{Z}^\ell} c_{w_0\alpha,\beta} \, \underline{x}^{-\mathrm{rev}(\beta)} \right)$$

$\mathrm{rev}(\beta) = (\beta_\ell, \dots, \beta_1)$

$$= \sum_{\beta\in\mathbb{Z}^\ell} c_{w_0\alpha,\beta} \, [[\underline{x}^{\mathrm{rev}(\beta)}]] \left(f \prod_{(i,j)\in\Delta_\ell^+} \left(1 - x_i/x_j\right)\right)$$

Cor: $\forall f \in \mathbb{Z}[\underline{x}^{\pm 1}]$

$$f = \sum_{\alpha,\beta\in\mathbb{Z}^\ell} c_{w_0\alpha,\beta} \, [[x^{\mathrm{rev}(\beta)}]] \left(f \prod_{(i,j)\in\Delta_\ell^+} \left(1 - x_i/x_j\right)\right) K_\alpha$$

$$\mathrm{poly}(f) = \sum_{\alpha,\beta\in\mathbb{Z}^\ell_{\geq 0}} c_{w_0\alpha,\beta} \, [[x^{\mathrm{rev}(\beta)}]] \left(f \prod_{(i,j)\in\Delta_\ell^+} \left(1 - x_i/x_j\right)\right) K_\alpha.$$