§ 4.1 $U_q(\mathfrak{g})$ - seminormal crystals

$\mathfrak{g}$ - sym. Kac-Moody Lie alg.; $U_q(\mathfrak{g})$.

I - nodes for Dynkin diagram.

P^* - coweight lattice $\{\alpha_i^\vee\} \subseteq P^*$

$P = \mathrm{Hom}_{\mathbb{Z}}(P^*, \mathbb{Z})$ weight lattice $\{\alpha_i\}_{i \in I} \subseteq P$.

Def: A $U_q(\mathfrak{g})$ - seminormal crystal is a set B:

- $wt : B \longrightarrow P$
- crystal ops: $f_i, e_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}$ $f_i(0) = e_i(0) = 0$

$\ni$ 1) $wt(e_i b) = wt(b) + \alpha_i$
$wt(f_i b) = wt(b) - \alpha_i$

2) $\varepsilon_i(b) = \max\{k \geq 0 \mid e_i^k(b) \neq 0\} < \infty$ $\underbrace{e_i^k(b) \to \cdots \to b}_{\varepsilon_i(b)} \underbrace{\to \cdots \to f_i^k(b)}_{\phi_i(b)}$

$\phi_i(b) = \max\{k \geq 0 \mid f_i^k(b) \neq 0\} < \infty$

$\langle \alpha_i^\vee, wt(b) \rangle = \phi_i(b) - \varepsilon_i(b)$

Def: A strict embedding of $B \longrightarrow B'$

is an injective map: $\Psi : B \cup \{0\} \longrightarrow B' \cup \{0\}$

$\Psi(0) = 0$ & Ψ commutes w/ $wt, f_i, e_i, \varepsilon_i, \phi_i$ $\forall i$

Note: This embedding maps B to a disjoint union of connected components of B'

Def: The tensor prod of $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$

w/ $wt(b_1 + b_2) = wt(b_1) + wt(b_2)$.

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i(b_1) \otimes b_2 & \varepsilon_i(b_1) > \phi_i(b_2) \\ b_1 \otimes e_i(b_2) & \varepsilon_i(b_1) \leq \phi_i(b_2) \end{cases}$$

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \varepsilon_i(b_1) \geq \phi_i(b_2) \\ b_1 \otimes f_i(b_2) & \varepsilon_i(b_1) < \phi_i(b_2) \end{cases}$$

Def: $\mathcal{O}_{int}$ = cat of $U_q(g)$-mods $M \ni M \simeq \bigoplus M_\lambda$

int. h.w. $U_q(g)$-mod.

(if we assume roots & coroots are lin indep).

Then for any $M \in \mathcal{O}_{int}$ has a <u>unique</u> crystal basis $(\mathcal{L}, B)$

$\Rightarrow$ yields a $U_q(g)$ - seminormal crystal.

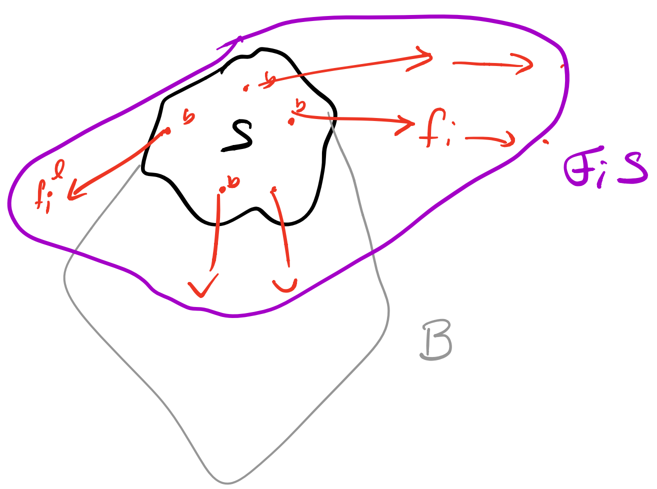
Def: the $U_q(g)$-crystal is the seminormal crystal that arises in this way

Def: The highest weight $U_q(g)$-crystal $B(\Lambda)$ is the crystal for the irred. h.w. module $V(\Lambda) \in \mathcal{O}_{int}$; $\Lambda \in P^+$

$\Rightarrow$ Any $U_q(g)$ crystal is a disjoint union of h.w. $U_q(g)$-crystals.

§.4.2 For $S \subseteq B$; $i \in I$ let

$$\mathcal{F}_i S := \{ f_i^m b \mid b \in S ; m \geq 0 \} \setminus \{0\} \subseteq B$$

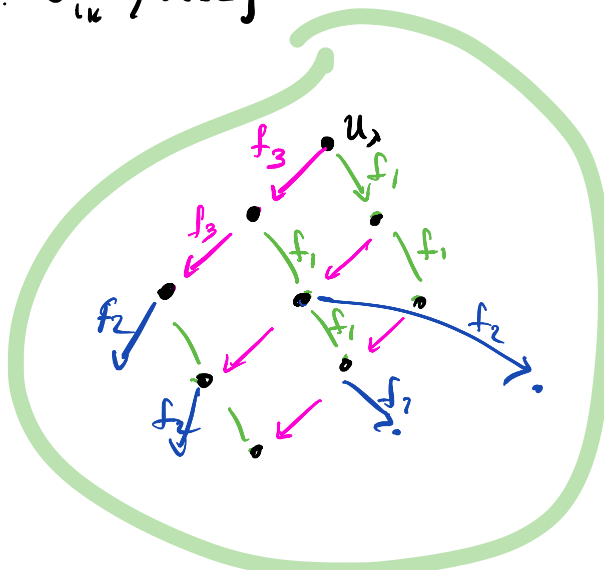


Let u_Λ the highest weight elmt of $B(\Lambda)$

<u>Definition:</u> A $U_q(g)$ - <u>Demazure crystal</u> is the subset of a h.w. $U_q(g)$ crystal $B(\Lambda)$ of the form.

$$\tilde{f}_{i_1} \dots\dots \tilde{f}_{i_k} \{u_\Lambda\}$$

<u>Ex:</u> $\tilde{f}_2 \tilde{f}_1 \tilde{f}_3 \{u_\lambda\}$



let $J \subseteq I$ $\quad \hat{P}^* \subseteq P^*$ $\ni$ $\{\alpha_i^\vee\}_{i\in J} \subseteq \hat{P}^*$

$\exists$ proj $z: P \longrightarrow \hat{P} := \mathrm{Hom}_{\mathbb{Z}}(\hat{P}^*, \mathbb{Z})$

Suppose $\{\alpha_i^\vee\}_{i\in I}$, $\{\alpha_i\}_{i\in I}$ and $\{z(\alpha_i)\}_{i\in J}$ are lin independent

<u>Def:</u> $U_q(g_J) \subseteq U_q(g)$ sub. alg. gen. $\{e_i, f_i\}_{i\in J}$ and $\{q^h\}_{h\in\hat{P}^*}$

<u>actually</u> a q. env. alg. for the cartan datum above.

Since (L, B) is a crystal basis for M as $U_q(g)$-mod and as a $U_q(g_J)$-mod. then:

<u>Def:</u> $\hat{B}$ = the associated $U_q(g_J)$-crystal of $\mathrm{Res}_{U_q(g_J)}(L, B)$

$\hat{B}$ is obtained by replacing $\quad wt \longmapsto z \circ wt$

and consider only $\{f_i, e_i\}_{i\in J}$

Denote $\vec{B} = \mathrm{Res}_J B$

Theorem: (4.1) For any $U_q(g)$ Demazure crystal S then its restriction $\mathrm{Res}_J S$ is isomorphic to a disjoint union of $U_q(g_J)$-Demazure crystals.

Proof: $S = \mathcal{F}_{i_1} \cdots\cdots \mathcal{F}_{i_k}\{u_\lambda\} \subseteq B(\Lambda) \leftarrow$ h.w. $U_q(g)$-crystal.

$$\begin{array}{ccccc} S & & & & B(\Lambda) \\ \downarrow & & & & \downarrow \\ \mathrm{Res}_J(S) & \subseteq & \mathrm{Res}_J(B(\Lambda)) & \simeq & \bigsqcup \text{ highest weight } U_q(g_J)\text{-crystals.} \end{array}$$

Since $\{\alpha_i^\vee\}_{i\in I}$ are lin independent then we can choose

$$\{\Lambda_j\}_{j\in I} \subseteq P \ni \langle \alpha_i^\vee, \Lambda_j\rangle = m\,\delta_{ij} \quad i,j\in I$$
$$m \in \mathbb{Z}_{\geq 1}$$

Set $\bar{J} = I \setminus J$ $\qquad P_{\bar{J}} := \sum_{i\in\bar{J}} \Lambda_i$

$$c := 1 + \max_i\{\varepsilon_i(b) \mid b\in S\}$$

Consider $B(cP_{\bar{J}})$ - a $U_q(g)$-crystal w/ h.w. $u_{cP_{\bar{J}}}$

$\forall b\in S.$

$\phi_i(u_{cP_{\bar{J}}}) > \varepsilon_i(b)$

since $\langle \alpha_i^\vee, wt(b)\rangle = \phi_i(b) - \varepsilon_i(b)$

$$\phi_i(u_{cP_{\bar{J}}}) = \langle \alpha_i^\vee, cP_{\bar{J}}\rangle = c\cdot m \qquad i\in\bar{J} \leftarrow$$

so $f_i(b\otimes u_{cP_{\bar{J}}}) = b\otimes f_i(u_{cP_{\bar{J}}}) \notin S\otimes\{u_{cP_{\bar{J}}}\}$

Recall **thm** (Joseph) $\forall \lambda \in P^+$; $\mu \in P^+ \cup \{\infty\}$

$\underbrace{B_w(\mu)}_{\text{Dem. crystal w/ h.w. } u_\mu} \otimes \{u_\lambda\}$ is a disjoint union of Demazure crystals

Note: $\uparrow$ in the original paper the $\otimes$-product rule is Backwards

$\Rightarrow$ the thm. holds for $u_\lambda \otimes B_w(\mu)$.

$$\Rightarrow \quad S \otimes \{u_{c\rho_J}\} = \bigsqcup \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{ b_r \otimes u_{c\rho_J} \}$$ for some $b_r \in S$.

and $i_j \in J$

$\Downarrow$

$e_i(b_r \otimes u_{c\rho_J}) = 0$

Now, $\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{ b_r \otimes u_{c\rho_J} \} = (\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{b_r\}) \otimes u_{c\rho_J}$ $\Downarrow$ $e_i(b_r) = 0$

(b/c $\phi_i(u_{c\rho_J}) \geq \varepsilon_i(b)$ $\forall b \in S$)

and $\phi_i(u_{c\rho_J}) = \langle \alpha_i^\vee, c\rho_J \rangle = 0$ $\forall i \in J$

So then $S = \bigsqcup \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{b_r\}$

where $\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} \{b_r\}$ are $U_q(\mathfrak{g}_J)$-Dem. crystals. $\blacksquare$

$e_i(b_r) = 0$. $\forall i \in J$.

$\boxed{\mathfrak{g} = \hat{\mathfrak{sl}}_\ell}$

Def: The extended affine Sym. group $\tilde{S}_\ell$

gen: $s_i \; (i \in I) \quad \tau$

rel'n's
$$s_i^2 = id$$
$$s_i s_j = s_j s_i \qquad j \neq i \pm 1$$
$$s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$$
$$\tau s_i = s_{i+1} \tau$$
$$\tau^\ell = id$$

Def: Affine Sym. group $\hat{S}_\ell \subseteq \tilde{S}_\ell$

gen: $\{s_i\}_{i \in I}$ w/ relns

let $\Sigma = \{\tau^i \mid i = 1, \dots, \ell\}$ $\qquad \tilde{S}_\ell = \hat{S}_\ell \rtimes \Sigma$

τ - aut. of Dynkin diag. $\quad i \longmapsto i+1$

$$\tau s_i \tau^{-1} = s_{\tau(i)}$$

Also see: $\quad \tilde{S}_\ell \subseteq GL(\mathfrak{h}^*)$

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \quad ; \lambda \in \mathfrak{h}^*$$
$$\tau(\Lambda_i) = \Lambda_{i+1}$$
$$\tau(\delta) = \delta$$

$$\underbrace{\{\Lambda_i\}_{i \in I} \cup \{\delta = \textstyle\sum_{i \in I} \alpha_i\}}_{\text{basis for } \mathfrak{h}^*}$$

$\Rightarrow \quad \Sigma \subseteq \tilde{S}_\ell \subseteq GL(\mathfrak{h}^*)$

$\rightarrow$ if $\sigma \in \Sigma$ then $\sigma(\alpha_i) = \alpha_{\sigma(i)}$
$\sigma(\delta) = \delta$ $\qquad$ and $\sigma(\rho) = \rho$

Def: 0-Hecke Monoid $\tilde{H}_e$ of $\tilde{S}_e$

Recall:
Monoid:
set w/ a map
$S \times S \to S$
① ass.
② has an identity.

gen: τ, s_i ; $i \in \mathbb{Z}$

relns: same as $\tilde{S}_e$ but w/ $s_i^2 = s_i$

Def: 0-Hecke Monoid H_e of S_e
w/ gen s_i $(i \neq 0)$ reln $s_i^2 = s_i$

length $w \in \hat{S}_e$ minimum $m \ni w = s_{i_1} \cdots s_{i_m}$
if $w = \tau^i v$, length(w) = length(v).

Def: $\sigma \in \Sigma$ and B, B' - $U_q(\hat{sl}_e)$-crystals.
A bijection $\theta : B \to B'$ is a σ-twist
if :
$$\sigma(\mathrm{wt}(b)) = \mathrm{wt}(\theta(b))$$
$$\theta(e_i(b)) = e_{\sigma(i)}\theta(b) \; ; \; \theta(f_i(b)) = f_{\sigma(i)} \quad \forall i \in \mathbb{Z}$$
$$\theta(0) = 0$$

view: $\tau \in \mathrm{Aut}(\text{Dynkin})$
$i \mapsto i+1$ $\quad \tau s_i \tau^{-1} = s_{\tau(i)}$.

Def: For any $\Lambda \in P^+$, $\exists$ unique twist (?)
$$F_\sigma^\Lambda : B(\Lambda) \longrightarrow B(\sigma(\Lambda)) \qquad \sigma(\alpha_i) = \alpha_{\sigma(i)}$$

$\Rightarrow$ $\theta_1 : B_1 \to B_1$ & $\theta_2 : B_2 \to B_2'$ both σ-twists.
then $\theta_1 \otimes \theta_2 : B_1 \otimes B_2 \to B_1' \otimes B_2'$ is a σ-twist.

Def: the σ-twist of h.w. $U_q(\hat{sl}_e)$-crystals

$$\mathcal{F}_\sigma^{\Lambda_1} \otimes \dots \otimes \mathcal{F}_\sigma^{\Lambda_p} : B(\Lambda_1) \otimes \dots \otimes B(\Lambda_p) \to B(\sigma(\Lambda_1)) \otimes \dots \otimes B(\sigma(\Lambda_p))$$

Def: let $\mathcal{D}(\hat{sl}_e)$ = set of all subsets $S \subseteq B \ni$

① B is a tensor prod. of h.w. $U_q(\hat{sl}_e)$-crystals.

② the image of S under map $B \xrightarrow{\sim} \bigsqcup_\Lambda B(\Lambda)$ to be a disjoint of $U_q(\hat{sl}_e)$-Demazure crystals.

Def: let $\mathcal{F}_\tau'$ be the operator $\mathcal{D}(\hat{sl}_e)$ that sends

$$\begin{array}{c} S \longmapsto \mathcal{F}_\tau'^{\Lambda_1} \otimes \dots \otimes \mathcal{F}_\tau'^{\Lambda_p}(S) \\ \cap \\ B(\Lambda^1) \otimes \dots \otimes B(\Lambda^p). \end{array}$$

Prop: The operators $\mathcal{F}_i$ $(i \in \mathbb{Z})$ & $\mathcal{F}_\tau$ send $U_q(\hat{sl}_e)$-Demazure crystals to $U_q(\hat{sl}_e)$-Demazure crystals.

$\Rightarrow$ $\mathcal{F}_i$, $\mathcal{F}_\tau$ are ops. on $\mathcal{D}(\hat{sl}_e)$ and satisfy the 0-Hecke relations for $\tilde{H}_e$.

Proof (Sketch)

Lemma: (Naoi) $\Lambda \in P^+$ $w \in W$

① $\tau B_w(\Lambda) \cong B_{\tau w \tau^{-1}}(\tau(\Lambda))$

② $\mathcal{F}_i(B_w(\Lambda)) = \begin{cases} B_w(\Lambda) & \text{if } \ell(s_i w) = \ell(w) - 1 \\ B_{s_i w}(\Lambda) & \text{if } \ell(s_i(w)) = \ell(w) + 1 \end{cases}$

③ $\forall w' \in W$ $\mathcal{F}_{w'} B_w(\Lambda)$ is well defined w/

$$\mathcal{F}_{w'} B_w(\Lambda) = B_{w''}(\Lambda) \text{ for some } w'' \in W.$$

<u>Def:</u> $w \in \tilde{H}_\ell$ define

$$\mathcal{F}'_w : \mathcal{D}(\hat{sl}_\ell) \longrightarrow \mathcal{D}(\hat{sl}_\ell)$$

$$\mathcal{F}_w := \mathcal{F}_{c_1} \cdots \mathcal{F}_{c_m} \qquad w = c_1 \cdots c_m$$
$$c_j \in \{s_i\}_{i \in I} \cup \{\tau\}$$

$H_\ell \curvearrowright \mathbb{Z}^\ell \qquad s_i \cdot \alpha = \begin{cases} s_i \alpha & \alpha_i > \alpha_{i+1} \\ \alpha & \alpha \leq \alpha_{i+1} \end{cases}$

<u>Def:</u> let $B^{gl}(\nu) =$ h.w. $U_q(gl_\ell)$-crystal w/ h.w. u_ν $\quad \nu \in \{\lambda \in \mathbb{Z}^\ell \mid \lambda_1 \geq \cdots \geq \lambda_\ell\}$

The Dem. crystal indexed by weak compositions. $\alpha \in \mathbb{Z}^\ell$

$\ni \quad \mathrm{sort}(\alpha) = p(\alpha) \cdot \alpha$

partition sorting of α — shortest word that sorts α to $\mathrm{sort}(\alpha)$ $\in \tilde{H}_\ell$

<u>Def:</u> $U_q(gl_\ell)$-Dem. crystal. index by α

$$BD(\alpha) := \mathcal{F}'_{p(\alpha)}\{u_{\mathrm{sort}(\alpha)}\} \subseteq B^{gl}(\mathrm{sort}(\alpha)).$$

$K_\alpha := \pi_{p(\alpha)} x^{\alpha^+}$ ← key polyn.

$\pi_i = \dfrac{x_i - x_{i+1} s_i}{x_i - x_{i+1}}$ ← satisfy the 0-Hecke relation.

$\pi_i K_\alpha = K_{s_i \cdot \alpha}$

Prop: The character of $U_q(\mathfrak{gl}_\ell)$-Dem. crystals H_ℓ are key polyn.

$$\mathrm{Char}_{\mathfrak{gl}}(BD(\alpha)) = \sum_{b \in BD(\alpha)}' x^{wt(b)} = K_\alpha(x_1, \dots, x_\ell)$$