Plan: — Interactions between π_i, Φ, poly

— Outline proof of Main Thm:

If $(\Psi;\gamma;w)$ is tame, then

$$\boxed{H(\Psi;\gamma;w) = \pi_w x_1^{\gamma_1} \Phi \pi_{\sigma(n_1)} x_1^{\gamma_2} \Phi \pi_{\sigma(n_2)} x_1^{\gamma_3} \cdots \Phi \pi_{\sigma(n_{\ell-1})} x_1^{\gamma_\ell}.}$$

Notations: $[d] = \{1,2,\dots,d\}$ $\qquad \Psi \subseteq \Delta_\ell^+$ root ideal

$\underline{x} = (x_1, x_2, \dots, x_\ell)$ $\qquad \epsilon_{(a,b)} = \epsilon_a - \epsilon_b$

$\underline{x}^{\pm 1} = (x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_\ell^{\pm 1})$

$\alpha \in \mathbb{Z}^\ell$, $\underline{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}$

$\sigma_1, \sigma_2, \dots, \sigma_{\ell-1}$: gens. of $\mathcal{H}_\ell$

$\sigma(d) = \sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_d$

$W_{[i,j)}$ = longest elt. in submonoid gen. by $\sigma_i, \sigma_{i+1}, \dots, \sigma_{j-1}$

$= (\sigma_{j-1}\sigma_{j-2}\cdots\sigma_i)(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\cdots(\sigma_{j-1}\sigma_{j-2})(\sigma_{j-1})$

$= (\sigma_i)(\sigma_i\sigma_{i+1})\cdots(\sigma_i\sigma_{i+1}\cdots\sigma_{j-1})$

$W_{\vec{a}} := W_{[a,\ell)}$

$= \sigma(a)\, W_{\overrightarrow{a+1}}$

Previous defs and facts

Demazure operators: $\pi_i(f) = \dfrac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}$

$\pi_w = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ if $w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}$

$\hat{\pi}_i = \pi_i - 1$, $\hat{\pi}_w$ def. similarly

Key polynomials: $K_\alpha = \pi_{p(\alpha)} \underline{x}^{\mathrm{sort}(\alpha)}$, $\alpha \in \mathbb{Z}^\ell$

$$\pi_i K_\alpha = \pi_{\sigma_i \alpha}$$

[Reiner-Shimozono] $\{K_\alpha \mid \alpha \in \mathbb{Z}^\ell_{\geq 0}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\underline{x}]$

[BMP] $\{K_\beta \mid \beta \in \mathbb{Z}^\ell\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\underline{x}^{\pm 1}]$

$$\mathrm{poly}(K_\alpha) = \begin{cases} K_\alpha, & \text{if } \alpha \in \mathbb{Z}^\ell_{\geq 0} \\ 0, & \text{if } \alpha \in \mathbb{Z}^\ell \setminus \mathbb{Z}^\ell_{\geq 0} \end{cases}$$

If $\hat{K}_\alpha = \sum_{\beta \in \mathbb{Z}^\ell} c_{\alpha,\beta} \underline{x}^\beta$

$$\Rightarrow f = \sum_{\alpha,\beta \in \mathbb{Z}^\ell} c_{w_0\alpha,\beta} \left(\text{coeff. of } \underline{x}^{\mathrm{rev}(\beta)} \text{ in } f \prod_{(i,j)\in\Delta^+_\ell} (1 - x_i/x_j)\right) K_\alpha$$

$$\mathrm{poly}(f) = \sum_{\alpha,\beta \in \mathbb{Z}^\ell_{\geq 0}} c_{w_0\alpha,\beta} \left(\text{coeff. of } \underline{x}^{\mathrm{rev}(\beta)} \text{ in } f \prod_{(i,j)\in\Delta^+_\ell} (1 - x_i/x_j)\right) K_\alpha$$

Nonsymmetric Catalan functions

$$H(\Psi; \gamma; w)(\underline{x}; q) = \pi_w \circ \mathrm{poly}\left(\prod_{(i,j)\in\Psi} (1 - q x_i/x_j)^{-1} \underline{x}^\gamma\right)$$

Recursion: If $\alpha \in \Psi$ is a removable root, then

$$H(\Psi; \gamma; w) = H(\Psi \setminus \alpha; \gamma; w) + q H(\Psi; \gamma + \epsilon_\alpha; w)$$

$n(\Psi) := (n_1, n_2, \ldots, n_{\ell-1})$, $n_i = \#$ boxes on or above main diagonal in row i that are not in Ψ.

$\sigma(d) := \sigma_{\ell-1}\sigma_{\ell-2}\cdots\sigma_d$

Right descent of w

$(\Psi;\gamma;w)$ is <u>tame</u> if $\{n(\Psi)_1+1, n(\Psi)_1+2, \ldots, \ell-1\} \subseteq \{i\in[\ell-1] \mid w\sigma_i = w\}$

$H(\Psi;\gamma;w)$ is <u>tame</u> if $(\Psi;\gamma;w)$ is tame.

Remark: if $(\Psi;\gamma;w)$ is tame, then $w = v\, w_{\overrightarrow{n(\Psi)_1+1}}$ is a length additive factorization for w

<u>Rotation operator</u>: Φ on $\mathbb{Z}[q,q^{-1}][\underline{x}^{\pm 1}]$

$x_i \mapsto x_{i+1}$, if $i \in [\ell-1]$

$x_\ell \mapsto q x_1$

$\boxed{\text{Main Thm}}$ If $(\Psi;\gamma;w)$ is tame and $\gamma_1 \geq 0$, then with notations above,

$$H(\Psi;\gamma;w) = \pi_w x_1^{\gamma_1} \Phi\, \pi_{\sigma(n_1)} x_1^{\gamma_2} \Phi\, \pi_{\sigma(n_2)} x_1^{\gamma_3} \cdots \Phi\, \pi_{\sigma(n_{\ell-1})} x_1^{\gamma_\ell}$$

$\boxed{\text{Prop 1}}$ ☆

1. $\forall f \in \mathbb{Z}[q][\underline{x}^{\pm 1}]$; $\forall i \in [\ell-1]$
   $\pi_i\,[\mathrm{poly}(f)] = \mathrm{poly}[\pi_i(f)]$
2. $\forall a \in \mathbb{Z}^{\ell}_{\geq 0}$, $\mathrm{poly}(\underline{x}^{\alpha}) = \underline{x}^{\alpha}$
3. Let $\gamma \in \mathbb{Z}^{\ell}$, $w \in \mathcal{H}_\ell$ be arbitrary
   $\exists k \in [\ell]$ s.t. $\sum_{a=k}^{\ell} \gamma_a < 0 \Rightarrow \mathrm{poly}(\underline{x}^{\gamma}) = 0$
4. $\exists k \in [\ell]$ s.t. $\sum_{a=k}^{\ell} \gamma_a < 0 \Rightarrow \forall$ root ideal Ψ
   $H(\Psi;\gamma, w) = 0$.
5. (Replacement)
   $\gamma_m = \gamma_{m+1} = \cdots = \gamma_\ell = 0 \Rightarrow \forall$ root ideals Ψ, Ψ' s.t.

$$\Psi \cap \Delta_m^+ = \Psi' \cap \Delta_m^+$$

$$H(\Psi; \gamma, w) = H(\Psi'; \gamma; w)$$

Proof sketch for ③, ④, ⑤ : Look at individual terms in $\prod_{(i,j) \in \Psi} (q x_i / x_j)^{d_{ij}}$

$\boxed{\text{Cor } 2}$: $H(\Psi; \gamma; w)$ lies in $\mathbb{Z}[\underline{x}][q]$ rather in $\mathbb{Z}[\underline{x}][[q]]$

i.e. $H(\Psi; \gamma; w)$ is a fin. lin. comb. of K_α, $\alpha \in \mathbb{Z}^\ell_{\geq 0}$ with coeff. in $\mathbb{Z}[q]$

$\boxed{\text{Prop } 3}$ ☆

$$\forall f \in \mathbb{Z}[q, q^{-1}][\underline{x}^{\pm 1}] \quad \forall i \in [\ell-2]$$

$$\pi_{i+1} \underline{\Phi}(f) = \underline{\Phi}[\pi_i(f)]$$

Furthermore as $\tau \sigma_i \tau^{-1} = \sigma_{i+1}$ in $\mathcal{H}_\ell$ $\forall i \in [\ell-2]$

$$\pi_{\tau v \tau^{-1}} \underline{\Phi}(f) = \underline{\Phi}\, \pi_v(f)$$

$$\forall v \in \mathcal{H}_{\ell-1} \times \mathcal{H}_1 \subseteq \mathcal{H}_\ell$$

$\boxed{\text{Lem } 4}$ ☆

$$\forall f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_{\ell-1}^{\pm 1}], \quad \forall a \geq 0$$

$$\text{poly}[x_1^a \underline{\Phi}(f)] = x_1^a \underline{\Phi}[\text{poly}(f)]$$

($x_1^a \underline{\Phi}$ commutes with poly if $a \geq 0$)

Sketch of proof: Verify on basis $\{K_\beta \mid \beta \in \mathbb{Z}^{\ell-1}\}$

$$K_\beta = \pi_{p(\beta)} \underline{x}^{\text{sort}(\beta)}$$

Use $\boxed{\text{Prop } 1}$ ① – ③, $\boxed{\text{Prop } 3}$.

Use $\boxed{\text{Prop 1}}$ ① ③ , $\boxed{\text{Prop 2}}$.

$\boxed{\text{Lem 5}}$ $\forall f, g \in \mathbb{Z}[\underline{x}^{\pm 1}]$ $\qquad \widehat{\pi_i} = \pi_i - 1$

① $x_{i+1}\, \pi_i(f) = \widehat{\pi_i}(x_i f)$, $\forall i \in [\ell-1]$

② $x_i^{-1}\, \pi_i(f) = \widehat{\pi_i}(x_{i+1}^{-1} f)$, $\forall i \in [\ell-1]$

③ $x_j^{-1}\, \pi_{j-1}(g) = \pi_{j-1}(x_{j-1}^{-1} g) + x_j^{-1} g$, $\forall j \in \{2, 3, \dots, \ell\}$

Sketch of proof: Straightforward computation for ①

② Multiply ① by $x_i^{-1} x_{i+1}^{-1}$, note that $\pi_i, \widehat{\pi_i}$ commutes with $x_i^{-1} x_{i+1}^{-1}$

③ Set $g = x_i f$, $j = i+1$ on ①

$\boxed{\text{Lem 6}}$ Let $f \in \mathbb{Z}[\underline{x}^{\pm 1}]$ be s.t. $s_i(f) = f$ $\forall a < i \leq \ell - 1$

$\Rightarrow$ $x_\ell\, \pi_{\ell-1} \pi_{\ell-2} \cdots \pi_a(f) = \widehat{\pi}_{\ell-1} \widehat{\pi}_{\ell-2} \cdots \widehat{\pi}_a(x_a f)$

$= \widehat{\pi}_{\ell-1}\, \pi_{\ell-2} \cdots \pi_a(x_a f)$

$\boxed{\text{Lem 7}}$ $\forall i \in [\ell]$, $\forall \alpha \in \mathbb{Z}^\ell$

$x_i^{-1} K_\alpha \in \operatorname{span}_{\mathbb{Z}} \{ K_\beta \mid \operatorname{sort}(\beta) = \operatorname{sort}(\alpha) - \epsilon_j \text{ for some } j \in [\ell] \}$.

Proof: Write $K_\alpha = \pi_v\, x^\mu$, $\mu = \operatorname{sort}(\alpha)$, $v = p(\alpha)$

Use $\boxed{\text{Lem 5}}$ ①, ② inductively on length(v)

$\boxed{\text{Lem 8}}$ $\forall f \in \mathbb{Z}[\underline{x}^{\pm 1}]$ $x_\ell \operatorname{poly}[x_\ell^{-1}\, \widehat{\pi}_{\ell-1}(f)] = \operatorname{poly}[\widehat{\pi}_{\ell-1}(f)]$.

Sketch of Proof: Verify on basis $\{x^\alpha \mid \alpha \in \mathbb{Z}^\ell_{\geq 0}\} \cup \{K_\beta \mid \beta \in \mathbb{Z}^\ell \setminus \mathbb{Z}^\ell_{\geq 0}\}$

Use $\boxed{\text{Prop 1}}$ ②, $\boxed{\text{Lem 7}}$

$\boxed{\text{Cor 9}}$ $\forall g \in \mathbb{Z}[\underline{x}^{\pm 1}]$, $\forall a \in [\ell - 1]$

☆

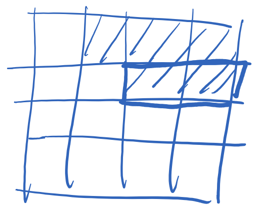
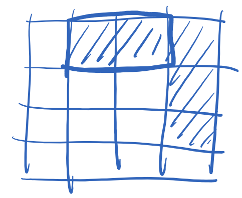
$$\mathrm{poly}[\pi_{W_{\overrightarrow{a+1}}}(x_a g)] = \mathrm{poly}[\pi_{\ell-2}\pi_{\ell-3}\cdots\pi_a\pi_{W_{\overrightarrow{a+1}}}(x_a g)] + x_\ell\,\mathrm{poly}[\pi_{W_{\vec{a}}}(g)]$$

Sketch: Rewrite identity $x_\ell\,\mathrm{poly}[\pi_{W_{\vec{a}}}(g)] = \mathrm{poly}[\hat{\pi}_{\ell-1}\pi_{\ell-2}\cdots\pi_a\pi_{W_{\overrightarrow{a+1}}}(x_a g)]$

use $\boxed{\text{Lem 6}}$, $\boxed{\text{Lem 8}}$.

Def: $\gamma \in \mathbb{Z}^\ell$, $R(\gamma) = (\gamma_2, \gamma_3, \dots, \gamma_\ell, 0)$

ψ root ideal, $R(\psi) = \{(i-1, j-1) \mid (i,j) \in \psi,\ i > 1\} \sqcup \{(i,\ell) \mid i \in [\ell-1]\}$

Eg: ψ:  $R(\psi)$: 

$\boxed{\text{Thm 10}}$ (Recursion on $H(\psi;\gamma;W_{\overrightarrow{a+1}})$)

Let $\gamma \in \mathbb{Z}^\ell$, ψ — root ideal

Set $a := n(\psi)_1$

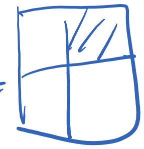
$$\gamma_1 \geq 0 \Rightarrow \boxed{H(\psi;\gamma;W_{\overrightarrow{a+1}}) = x_1^{\gamma_1}\,\Phi\left[H(R(\psi);R(\gamma);W_{\vec{a}})\right]}$$

$\boxed{\text{Rem 11}}$ Last column in $R(\psi)$ is added to get the right nonsymm. Catalan function of length ℓ.

$$R(\gamma)_\ell = 0 \xRightarrow{\boxed{\text{Prop 1}}\ 5} H(R(\psi);R(\gamma);W_{\vec{a}}) = H(\psi';R(\gamma);W_{\vec{a}})$$

for all $\psi' \subseteq \Delta^+_\ell$ s.t. $\psi' \cap \Delta^+_{\ell-1} = R(\psi) \cap \Delta^+_{\ell-1}$

$\boxed{\text{Eg. 12}}$ $\quad \ell = 2,\ \gamma = (3,2),\ \psi = \Delta_2^+$

$\Rightarrow a = 1,\ W_{\vec{a+1}} = 1,\ W_{\vec{a}} = \sigma_1 \quad \psi =$  $= R(\psi)$

$$H(\Delta_2^+; \gamma; W_{\vec{a+1}}) = \mathrm{poly}\left[(1 - q x_1/x_2)^{-1} x_1^3 x_2^2\right]$$
$$= \mathrm{poly}\left(x_1^3 x_2^2 + q x_1^4 x_2 + q^2 x_1^5 + q^3 x_1^6 x_2^{-1} + \dots\right)$$
$$= x_1^3 x_2^2 + q x_1^4 x_2 + q^2 x_1^5$$

$$x_1^{\gamma_1} \Phi\left[H(R(\Delta_2^+); R(\gamma); W_{\vec{a}})\right] = x_1^3\, \Phi\left[\pi_1\, \mathrm{poly}\left[(1 - q x_1/x_2)^{-1} x_1^2\right]\right]$$
$$= x_1^3\, \Phi\left[\pi_1 (x_1)^2\right]$$
$$= x_1^3\, \Phi\left(x_1^2 + x_1 x_2 + x_2^2\right)$$
$$= x_1^3 \left(x_2^2 + q x_1 x_2 + q^2 x_1^2\right)$$
$$= q^2 x_1^5 + q x_1^4 x_2 + x_1^3 x_2^2$$

Outline of proof for <u>Main Thm</u>

To show: $H(\psi; \gamma; w) = \pi_w x_1^{\gamma_1} \Phi \pi_{\sigma(n_1)} x_1^{\gamma_2} \Phi \pi_{\sigma(n_1)} x_1^{\gamma_3} \dots \Phi \pi_{\sigma(n_{\ell-1})} x_1^{\gamma_\ell}$.

Induction on m, min. index $\gamma_m = \gamma_{m+1} = \dots = \gamma_\ell = 0$
$(m = \ell + 1$ if $\gamma_\ell \neq 0)$.

<u>Base case</u> $(m=1)$: $\quad \gamma = 0 \Rightarrow H(\Psi; \gamma; w) = 1$ $\quad \boxed{\text{Prop 1}}$ (5)

<u>Inductive step</u> (Assume $m > 1$)

As $(\Psi; \gamma; w)$ is tame, $w = v\, w_{\overrightarrow{n_1+1}}$ for some $v \in \mathcal{H}_\ell$

$$\Rightarrow H(\Psi; \gamma; w) = \pi_v H(\Psi; \gamma; w_{\overrightarrow{n_1+1}})$$

$$= \pi_v x_1^{\gamma_1} \left[H(R(\Psi); R(\gamma); w_{\vec{n_1}}) \right] \quad \boxed{\text{Thm 10}}$$

$$= \pi_v \underbrace{x_1^{\gamma_1} \Phi\, \pi_{w_{\vec{n_1}}}} x_1^{\gamma_2} \Phi\, \pi_{\sigma(n_2)} x_1^{\gamma_3} \cdots \Phi\, \pi_{\sigma(n_{\ell-1})} x_1^{\gamma_\ell} \underbrace{\Phi\, \pi_{\sigma(1)} x_1^0}_{1}$$

$$= \underbrace{\pi_v \pi_{w_{\vec{n_1}}}}_{\pi_w} x_1^{\gamma_1} \Phi\, \pi_{\sigma(n_1)} x_1^{\gamma_2} \Phi\, \pi_{\sigma(n_2)} x_1^{\gamma_3} \cdots \Phi\, \pi_{\sigma(n_{\ell-1})}$$

$$x_1^{\gamma_1} \Phi\, \pi_{w_{\vec{n_1}}} = x_1^{\gamma} \Phi\, \pi_{w_{[n,\ell-1)}} \pi_{\sigma(n_1)} = \pi_{w_{\vec{n_1}}} x_1^{\gamma_1} \Phi\, \pi_{\sigma(n_1)}$$

$$w_{[n,\ell)} = w_{[n,\ell-1)} \sigma(n_1)$$

$\boxed{\text{Prop 3}}$ and $x_1^{\gamma_1}$ is symm. by $\pi_{[n,\ell-1)}$