Informal ADM seminar 10/12

Monday, October 12, 2020 1:53 PM

[BMPS2, Section 4]

Schur functions

$\gamma = (\gamma_1 \ldots \gamma_\ell)$ $\quad h_k$ = complete symmetric functions

$h_0 = 1$

Def $S_\gamma = \det(h_{\gamma_i + j - i})$

Ex $\gamma = (3,1,2,5)$ $\quad S_\gamma = \begin{vmatrix} h_3 & h_4 & h_5 & h_6 \\ 1 & h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 & h_3 \\ h_2 & h_3 & h_4 & h_5 \end{vmatrix} = 0$

same!

h_{γ_i} on diag

Ex $\gamma = (4,7,1,6)$

$$S_\gamma = \begin{vmatrix} h_4 & h_5 & h_6 & h_7 \\ h_6 & h_7 & h_8 & h_9 \\ 0 & 1 & h_1 & h_2 \\ h_3 & h_4 & h_5 & h_6 \end{vmatrix} = (-1)^2 \begin{vmatrix} h_6 & h_7 & h_8 & h_9 \\ h_4 & h_5 & h_6 & h_7 \\ h_3 & h_4 & h_5 & h_6 \\ 0 & 1 & h_1 & h_2 \end{vmatrix}$$

$$= S_{(6,5,5,2)}$$

Lemma $\rho = (\ell-1, \ell-2, \ldots 0)$, then

$$S_\gamma(x) = \begin{cases} \operatorname{sgn}(\gamma+\rho) \cdot S_{\operatorname{sort}(\gamma+\rho) - \rho}(x) & \text{if } \gamma+\rho \text{ has distinct} \geq 0 \text{ parts} \\ 0, & \text{otherwise.} \end{cases}$$

Ex $\gamma = (3,1,2,5)$ $\quad \rho = (3,2,1,0)$ $\quad \gamma+\rho = (6,3,3,5)$

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$\gamma = (4,7,1,6)$ $\gamma+\rho = (7,9,2,6)$

Proof: We see the entries of $\gamma+\rho$ in the last column of the matrix defining S_γ.

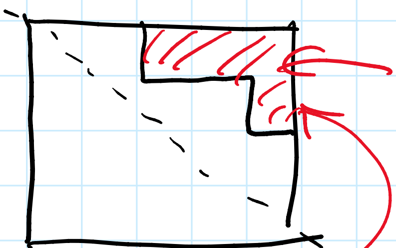
sorting $\gamma+\rho \longleftrightarrow$ permuting the rows in the matrix.

$\pi : \mathbb{Q}[z_1^{\pm 1} \dots z_\ell^{\pm 1}] \longrightarrow \Lambda$ linear map defined by $\pi(z^\gamma) = S_\gamma$ for all γ.

2 Catalan functions

$\gamma = (\gamma_1 \dots \gamma_\ell)$

Ψ = root ideal =



Young diagram above the diagonal

$\{(1,3),(1,4),(1,5),(2,5)\} \subset \Delta^+ = \{(i,j) \mid i<j\}$

Catalan fn: $\boxed{H(\Psi,\gamma) = \pi\Big(\prod_{(i,j)\in\Psi} (1 - t\, z_i/z_j)^{-1} z^\gamma\Big)}$

Can expand this as a power series

$(1 - t\, z_i/z_j)^{-1} = 1 + t\, z_i \dots + t^2 z_i^2 \dots$

$$(1-t\,z_i/z_j)^{-1} = 1 + t\frac{z_i}{z_j} + t^2\frac{z_i^2}{z_j^2} + \dots$$

$\rightsquigarrow$ infinite sum of monomials in z $\rightsquigarrow$

$\rightsquigarrow$ applying π changes each of them to a Schur.

Example 4.5. With $\ell = 4$, $\mu = 3321$, and $\Psi = \{(1,3),(2,4),(1,4)\}$, we have

$$\begin{aligned} H(\Psi;\mu) &= (1-tR_{13})^{-1}(1-tR_{24})^{-1}(1-tR_{14})^{-1}s_{3321} \\ &= s_{3321} + t(s_{3420}+s_{4311}+s_{4320}) + t^2(s_{4410}+s_{5301}+s_{5310}) \\ &\quad + t^3(s_{63-11}+s_{5400}+s_{6300}) + t^4(s_{64-10}+s_{73-10}) \\ &= s_{3321} + t(s_{4320}+s_{4311}) + t^2(s_{4410}+s_{5310}) + t^3 s_{5400}. \end{aligned}$$

Proposition 4.1 is used to truncate the series to terms s_α with $\alpha+\rho \in \mathbb{Z}^\ell_{\geq 0}$ for the second equality and it is used again for the third to give $s_{3420} = s_{5301} = s_{64-10} = s_{73-10} = 0$ and $s_{63-11} = -s_{6300}$.

$$H(\Psi;\mu) = \pi\left((1-t\,z_1/z_3)^{-1}(1-t\,z_2/z_4)^{-1}(1-t\,z_1/z_4)^{-1}z^\mu\right)$$

$$= \pi\left[z^\mu + t\left[\frac{z_1}{z_3}z^\mu + \frac{z_2}{z_4}z^\mu + \frac{z_1}{z_4}z^\mu\right] + \dots\right]$$

$$= S_{3321} + t(S_{4311} + S_{3420} + S_{4320}) + \dots$$

In principle, infinite series in t, coefs = compositional Schurs.

<u>Fact</u> In fact, only finitely many terms are nonzero.

(3) Hall–Littlewood polynomials.

$$\boxed{H_m = \sum_{i,j\geq 0} (-1)^i t^j h_{m+i-j}\, e_i^\perp h_j^\perp} = \text{operator}$$

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$(m \in \mathbb{Z})$ on symmetric functions

[Jing / Garsia HL operator].

$e_i^\perp, h_j^\perp$ are adjoint to multiplication by e_i, h_j under standard bilinear form on Λ where s_λ are orthonormal.

Note: $e_i^\perp, h_j^\perp$ lower the degree (resp. by i and j)

$\Rightarrow$ H_m has finitely many terms when applied to a given function.

denoted $\mathbf{H}_\gamma$ in paper

$$H_\gamma := H_{\gamma_1} \cdots H_{\gamma_\ell}$$

(Compositional) HL polynomial $= \boxed{H_\gamma(1)}$

Prop: Hall-Littlewoods are special case of Catalan

Catalan $H(\Delta^+, \gamma) = H_\gamma(1)$.

$\Psi = \Delta^+$ Proof: Refer to Zabrocki / Shimozono.

Another operator $\Phi : \mathbb{Q}[z_1^{\pm} \ldots z_\ell^{\pm}] \to \Lambda$

$$\Phi(z^\gamma) = H_\gamma(1) =$$

$$= \pi\big(\prod \quad (1 - t\, z_i/z_j \ldots z^\gamma)$$

$$= \pi\Big(\prod_{(i,j)\in\Delta^+}(1 - t\, z_i/z_j)^{-1} z^\gamma\Big)$$

<u>Observation</u>: $\Phi(-) = \pi\Big(\prod_{(i,j)\in\Delta^+}(1 - t\, z_i/z_j)^{-1}\cdot -\Big)$

<u>Lemma</u> $H(\Psi,\gamma) = \Phi\Big(\prod_{(i,j)\in\Delta^+\setminus\Psi}(1 - t\, z_i/z_j)\, z^\gamma\Big)$

Why this is nice? This is a finite polynomial in z's (no inverses!)

$\Rightarrow H(\Psi,\gamma) =$ finite sum of $H_\gamma(1)$.
of $2^{|\Delta^+\setminus\Psi|}$ terms.

<u>Proof</u> $H(\Psi,\gamma) = \pi\Big(\prod_{(i,j)\in\Psi}(1 - t\, z_i/z_j)^{-1} z^\gamma\Big)$

$$= \pi\Big(\prod_{(i,j)\in\Delta^+}(1 - t\, z_i/z_j)^{-1}\cdot \prod_{(i,j)\in\Delta^+\setminus\Psi}(1 - t\, z_i/z_j)\cdot z^\gamma\Big)$$

$=$ (by observation) $= \Phi(\qquad)$ $\square$.

We have identities for operators:

<u>Facts</u>: (a) $e_d^\perp H_m = H_m e_d^\perp + H_{m-1} e_{d-1}^\perp$

[commute $[e_d^\perp, H_m]$ by definition of H_m, note that $e_d^\perp$ commutes with $e_i^\perp, h_j^\perp$]

(b) $e_d^\perp H_\gamma = e_d^\perp H_{\gamma_1}\cdots H_{\gamma_\ell} =$

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$$= H_{\gamma_1} e_d^\perp H_{\gamma_2} \cdots H_{\gamma_\ell} + H_{\gamma_1 - 1} e_{d-1}^\perp H_{\gamma_2} \cdots H_{\gamma_\ell}$$

$$= \cdots$$

$$= \sum_{\substack{S \subset \{1..\ell\} \\ |S| \le d}} H_{\gamma - \epsilon_S}\, e^\perp_{d-|S|}$$ (by induction from (a))

where ϵ_S = char. function of S.

(c) $$e_d^\perp H_\gamma(1) = \sum_{\substack{S \subset \{1..\ell\} \\ |S| \le d}} H_{\gamma - \epsilon_S}\, e^\perp_{d-|S|}(1)$$

zero unless $|S| = d$

$$= \sum_{\substack{S \subset \{1..\ell\} \\ |S| = d}} H_{\gamma - \epsilon_S}(1)$$ [≈ "dual Pieri rule" for HL]

(d) $$e_d^\perp \Phi(z^\gamma) = e_d^\perp H_\gamma(1) = \sum_{|S|=d} H_{\gamma - \epsilon_S}(1)$$

$$= \sum_{|S|=d} \Phi(z^{\gamma - \epsilon_S})$$

$\Rightarrow$ for any argument

$$\boxed{e_d^\perp\, \Phi(\,\text{—}\,) = \sum_{|S|=d} \Phi(z^{-\epsilon_S} \cdot \text{—})}$$

<u>Lemma</u> $$\boxed{e_d^\perp(H(\Psi, \gamma)) = \sum_{|S|=d} H(\Psi, \gamma - \epsilon_S)}$$

$|S|=d$

We will need this in the future

Proof By the above:

$$H(\Psi,\gamma) = \Phi\Big(\prod_{(i,j)\in\Delta^+\setminus\Psi}(1-t\,{}^{z_i}/_{z_j})\cdot z^\gamma\Big)$$

$$e_d^\perp H(\Psi,\gamma) = \sum_S \Phi\Big(z^{-\epsilon_S}\cdot\prod_{(i,j)\in\Delta^+\setminus\Psi}(1-t\,{}^{z_i}/_{z_j})\cdot z^\gamma\Big)$$

$$= \sum_S H(\Psi,\gamma-\epsilon_S).$$

Lemma Suppose that $\gamma_\ell = 0$. Then

$$\boxed{H(\Psi,\gamma) = H(\tilde\Psi,\tilde\gamma)}$$

where $\tilde\gamma = (\gamma_1 \dots \gamma_{\ell-1})$, and $\tilde\Psi$ = truncation of Ψ to $(\ell-1)\times(\ell-1)$ square

(i.e. $\tilde\Psi$ is obtained from Ψ by forgetting the last column).

Proof $H(\Psi,\gamma) = \Phi\Big(\prod_{(i,j)\in\Delta^+\setminus\Psi}(1-t\,{}^{z_i}/_{z_j})\, z_1^{\gamma_1}\cdots z_{\ell-1}^{\gamma_{\ell-1}}\,z_\ell^0\Big)$

Note that z_ℓ appears in non positive degrees here $\left({}^{z_i}/_{z_\ell}\right)$ $\rightsquigarrow$ the sum will include

$H_{\mu_1\dots\mu_\ell}(1)$ with $\mu_\ell \leq 0$

$$H_{\mu_1\dots\mu_\ell}(1) = H_{\mu_1}H_{\mu_2}\cdots\underbrace{H_{\mu_\ell}(1)}_{\shortparallel}$$

$$H_{\mu_1 \cdots \mu_\ell}(1) = H_{\mu_1} H_{\mu_2} \cdots \underbrace{H_{\mu_\ell}(1)}_{\substack{\| \\ 0}}$$

unless $\mu_\ell \geq 0$

$$\Rightarrow H_{\mu_1 \cdots \mu_\ell}(1) = \begin{cases} H_{\mu_1 \cdots \mu_{\ell-1}}(1) & \text{if } \mu_\ell = 0 \\ 0 & \text{if } \mu_\ell < 0 \end{cases}$$

$\Rightarrow$ What's left is the sum over monomials with $\mu_\ell = 0 \Rightarrow \Phi\left(\prod_{\Delta^+_{\ell-1} \setminus \Psi} (1 - t^{z_i}/z_j)\, z^{\tilde{\gamma}}\right)$

$= H(\tilde{\Psi}, \tilde{\gamma})$.

ignore all $1 - t^{z_i}/z_\ell$, $i < \ell$.

<u>Rmk</u> In the paper:

$$\Phi\left((1 - t^{z_i}/z_j) - \right) = (1 - t R_{ij}) \Phi(-)$$

need to be careful with R_{ij} $\rightarrow R_{ij}(H_\gamma) = H_{\gamma + \epsilon_i - \epsilon_j}$