Adm-2020_10_19-presentation

Monday, October 19, 2020 8:24 AM

Section 5: Recurrences for the Catalan Functions BMPS2 1804.03701

Recall $$H(\overset{\swarrow\text{ root ideal}}{\Psi};\gamma) = \prod_{(i,j)\in\Delta^+\setminus\Psi}\left(1-t\frac{z_i}{z_j}\right)H_\gamma$$

Ψ is root ideal $\gamma\in\mathbb{Z}^\ell$ $\frac{z_i}{z_j}H_\gamma = H_{\gamma+\epsilon_i-\epsilon_j}$

In Section 5, Prove Propn 5.6

$$① \quad H(\Psi;\mu) = H(\Psi\cup\beta;\mu) - tH(\Psi\cup\beta;\underset{\mu+\epsilon_k-\epsilon_\ell}{\underset{\shortparallel}{\mu+\epsilon_\beta}})$$

$\Psi\cup\underset{(k,\ell)}{\underset{\shortparallel}{\beta}}$ = ideal

$$② \quad H(\Psi;\mu) = H(\Psi\setminus\alpha;\mu) + tH(\Psi;\mu+\epsilon_\alpha)$$

$\Psi\setminus\alpha$ = ideal ① ⇒ ②

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① ⇒ ② ⇒ Cor 5.7

$\beta=(k\ell)$

$\Psi\cup\beta$ = ideal

Proof: $$H(\Psi,\mu) = \prod_{(ij)\in\Delta^+\setminus\underset{\Psi\beta}{\Psi}}\left(1-t\frac{z_i}{z_j}\right)H_\mu$$

$$\overset{\beta\text{ term}}{\underline{\left(1-t\frac{z_k}{z_\ell}\right)}}\prod_{(ij)\in\Delta^+\setminus(\Psi\cup\beta)}\left(1-t\frac{z_i}{z_j}\right)H_\mu$$

$$\prod_{(ij)\in\Delta^+\setminus(\Psi\cup\beta)}\left(1-t\frac{z_i}{z_j}\right)H_\mu - t\prod_{(ij)\in\Delta^+\setminus(\Psi\cup\beta)}\left(1-t\frac{z_i}{z_j}\right)\underbrace{\frac{z_k}{z_\ell}H_\mu}_{H_{\mu+\epsilon_k-\epsilon_\ell}}$$

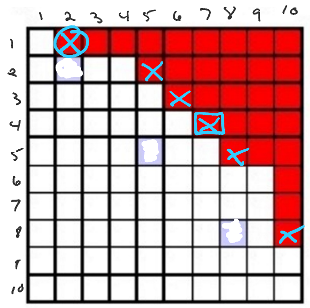
$$= H(\Psi\cup\beta;\mu) - tH(\Psi\cup\beta;\mu+\epsilon_\beta) \quad \boxed{3}$$

Definitions (to get to notation of Cor 5.7)

removable $\alpha \in \Psi$ is removable

$\Psi \setminus \alpha$ is an ideal

$(1,2)$ $\text{down}_\Psi(1) = 2$ $\text{down}_\Psi(4) = 7$



(12)

(25)

(36)

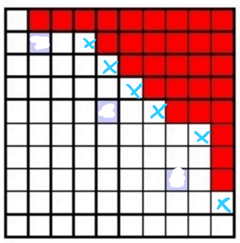
(58)

(47)

(8 10)

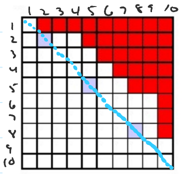
addable $\beta \in \Delta^+ \setminus \Psi$ is addable if

$\Psi \cup \beta$ is an ideal



down_\psi(x) Ψ if (x,j) is removable

$\text{down}_\Psi(x) = j$ else undefined



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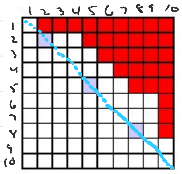
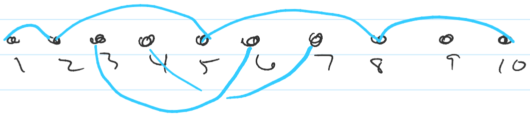
up_psi(x) if $\alpha = (i\ x)$ is removable

$\text{up}_\Psi(x) = i$ else undefined

$\text{up}_\Psi(7) = 4$



bounce graph



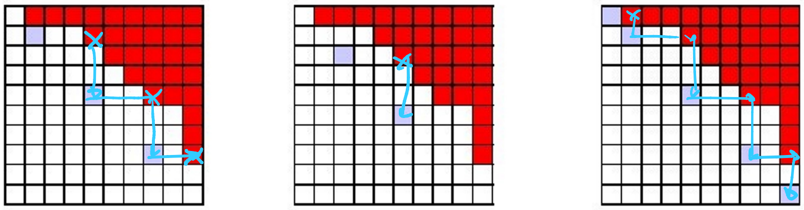
$1 \dots \ell$

edges $(r, \text{down}_\Psi(r))$

bounce paths

$$B_\Psi(a,b) := |\,\text{path}_\Psi(a,b)| - 1 = m.$$

Example 5.3. Examples of downpath, uppath, and bounce for the root ideal Ψ below:



$\text{path}_\Psi(2,8) = 2, 5, 8$ $\text{downpath}_\Psi(3) = 3, 6$ $\text{uppath}_\Psi(10) = 10, 8, 5, 2, 1$

$$B_\Psi(2,8) = 2,\ B_\Psi(1,10) = 4,\ B_\Psi(3,6) = 1, \text{ and } B_\Psi(3,3) = 0.$$

ex l

bot$_\psi$(r) = max elt of bounce path containing r

top$_\psi$(r) = min ...

$\text{bot}_\psi(5) = 10$

$\text{top}_\psi(5) = 1$

path$_\psi$(a,b) =

$(a, \text{down}_\psi(a), \text{down}_\psi^2(a), \ldots b)$

$\text{path}_\psi(2,8) = 2, 5, 8$

$\text{path}_\psi(1,10) = 1, 2, 5, 8, 10$

downpath$_\psi$ (r) $= \text{path}_\psi(r, \text{bot}_\psi(r))$

$\text{downpath}_\psi(5) = \text{path}_\psi(5\ 10) = 5, 8, 10$

$\text{uppath}_\psi(5) = 5, 2, 1$

$\text{uppath}_\psi(r) = \text{reversed}\ (r, \text{top}_\psi(r))$

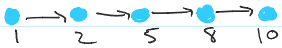
bounce a to b : B$_\psi$(a, b) $= |\text{path}_\psi(ab)| - 1$

$a \leq b$

$B_\psi(2,8) = |\{2 5 8\}| - 1 = 2$

$B_\psi(2,2) = 0$

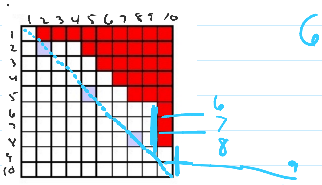
$B_\psi(1,10) = 4$



Definitions not used again in this section

wall in rows r, r+1

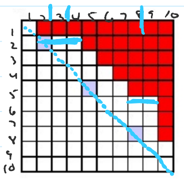
if row r & r+1 in ψ
have same length



6,7 and 7,8 and 9,10

ceiling in columns c, c+1

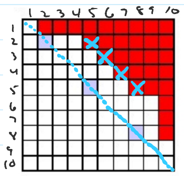
If columns c, c+1 have
same length



2,3 and 3,4 and 8,9

mirror in rows r, r+1

If have removable roots
(r, c) and (r+1, C+1)

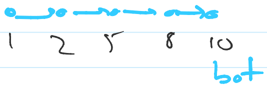


2,3 and 3,4 and 4,5

See Ex 5.5.

(2,5)
(3,6)
(4,7)
(5,8)

Back to Cor 5.7 and its proof



For $\underline{m=2}$, $\text{downpath}_\psi(2) = 2, 5, 8, 10$

$\underline{\text{ex of Cor 5.7}}$

$$H(\psi;\mu) = \underset{\substack{\wedge\\ t^0}}{H}(\psi\setminus(2,5);\underset{\substack{+\\ \epsilon_2-\epsilon_2}}{\mu}) + t\,H(\psi\setminus(5,8);\mu+\epsilon_2-\epsilon_5)$$

$$+\ t^2 H(\psi\setminus(8,10);\mu+\epsilon_2-\epsilon_8) + t^3 H(\psi;\mu+\epsilon_2-\epsilon_{10})$$

Cor 5.7/ Fix $1 \le m \le l$

$$H(\psi;\mu) = \sum_{z \in \text{downpath}_\psi(m)} t^{B_\psi(m,z)} H(\psi^z;\mu+\epsilon_m-\epsilon_z)$$

where $\psi^z = \psi\setminus(z, \text{down}_\psi(z))$ $\quad z \ne \text{bot}_\psi(m)$

$\psi^{\text{bot}_\psi(m)} = \psi.$

Pf/ Induct on $|\text{downpath}_\psi(m)|$. If $= 1$ RHS has 1 term and is at $z = \underset{\substack{\shortparallel\\ \text{bot}_\psi(m)}}{m}$

$B_\psi(m,m) = 0$

If > 1 let $m' = \text{down}_\psi(m) \ne m$

Note $B_\psi(m',z) + 1 = B_\psi(m,z)$ if $z \in \text{downpath}_\psi(m')$

By 5.6

$$H(\psi;\mu) = H(\overset{\substack{\psi\setminus(m,m')\\ \shortparallel}}{\psi^m};\mu) + t\,H(\psi;\mu+\epsilon_m-\epsilon_{m'})$$

$$= \underline{H(\psi^m;\mu)} + \underset{\substack{\nearrow\\ \text{induction}}}{t^{1}}\Big[\sum_{z \in \text{downpath}_\psi(m')} t^{B_\psi(m',z)} H(\psi^z;(\mu+\underline{\epsilon_m-\epsilon_{m'})+\epsilon_{m'}-\epsilon_z}\Big]$$

$$= \sum_{z \in \text{downpath}_\psi(m)} t^{B_\psi(m,z)} H(\psi^z;\mu+\underline{\epsilon_m-\epsilon_z})$$

∎