Catalan functions and k-Schur functions

(1) Chen-Haiman conjecture r.c.

$$\mathfrak{s}^{(k)}_\lambda(x,t) = H(\Delta^k(\mu);\mu).$$

$$\mathfrak{s}^{(k)}_\lambda(x;t) = s^{(k)}_\lambda(x;t)$$

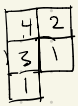
(2) k-Schur Branching: We can express k-Schur functions $s^{(k)}_\mu(x;t)$ in terms of $(k+1)$-Schur functions, with coefficients in $\mathbb{N}[t]$.

Preliminaries:

Par^k_ℓ = length ℓ partitions, $\mu_i \le k$.

Par^k = any partition, $\mu_i \le k$.

Hook length.



$(k+1)$-core: A partition with no box of hook length equal to $k+1$.

Thm: There exists a bijection p from $(k+1)$-cores to Par^k. If κ is a $(k+1)$-core then $p(\kappa)$ is the partition whose rth row λ_r is the number of boxes in the r-th row of κ having hook length $\le k$.

4-core = (3+1)-core

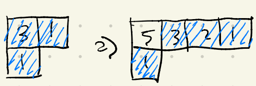
$\kappa = (4,1)$ $\qquad p(\kappa) = (3,1)$

Strong Cover: A strong cover $\tau \Rightarrow \kappa$ is a pair of $(k+1)$-cores s.t. $\tau \subseteq \kappa$ and $|p(\tau)| + 1 = |p(\kappa)|$

A strong marked cover $\tau \overset{r}{\Rightarrow} \kappa$ is a strong cover with a positive integer r which is allowed to be the smallest row index of any connected comp. of the skew shape κ/τ.

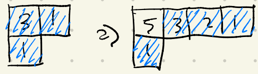
e.g. 4-cores

$(21) \Rightarrow (41)$ $\qquad p((21))$



e.g. 4-cores

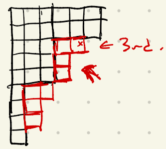
$(21) \Rightarrow (41)$



$p(21) = (21)$ $\qquad p(41) = (31)$

e.g. 5-cores, $\tau = 663331111$, $\kappa = 665443221$

$\tau \subset \kappa$.



$p(\tau) = 332221111$ $\qquad p(\kappa) = 222222221$

$$\tau \overset{3}{\Rightarrow} \kappa$$

Let $\eta = (\eta_1, \eta_2, \ldots) \in \mathbb{Z}^\infty_{\ge 0}$, $|\eta| = m$ finite.
Let $v_i = \eta_1 + \cdots + \eta_{i-1}$ $(v_1 = 0,\ v_2 = \eta_1,\ v_3 = \eta_1 + \eta_2, \ldots)$.

A strong marked tableaux (SMT) T of weight η is a sequence of strong marked covers

$$\kappa^{(0)} \overset{r_1}{\Rightarrow} \kappa^{(1)} \overset{r_2}{\Rightarrow} \kappa^{(2)} \overset{r_3}{\Rightarrow} \cdots \overset{r_m}{\Rightarrow} \kappa^{(m)}$$

s.t. $r_{v_i+1} \ge r_{v_i+2} \ge \cdots \ge r_{v_i+\eta_i} = r_{v_{(i+1)}}$

A vertical SMT (VSMT) T of weight η: defined the same, except

$$r_{v_i+1} < r_{v_i+2} < \cdots < r_{v_{(i+1)}}$$

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Spin: The spin of a strong marked cover $\tau \Rightarrow \kappa$, is defined as $c(h-1)+N$, where

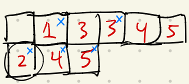
- c: the number of connected components of κ/τ
- h: the height of each component
- N: the number of components entirely contained in rows $> r$.

If we have an SMT T, then spin(T) as the sum of all spins

$(1) \Rightarrow (2) \Rightarrow (2\,1) \Rightarrow (4\,1) \Rightarrow (5\,2) \Rightarrow (6\,3)$

$p((1))$, $P((2))$, $P((2\,1))$, ...

(1) (2) $(2\,1)$ $(3\,1)$, $(3\,2)$ $(3\,3)$



$r_1 = 1$

$r_2 = 2$ $\qquad \eta = (1,2,2)$

inside$(T) = p(\kappa^{(0)})$ (e.g. (1))

outside$(T) = p(\kappa^{(m)})$ (e.g. (3 3))

For $f \in \Lambda$, let $f^\perp$ be the linear operator on Λ that is adjoint w.r.t. the Hall inner product.

$$\langle f^\perp(g), h\rangle = \langle g, f\cdot h\rangle$$

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$\mathfrak{s}_\mu^{(k)}$, $s_\mu^{(k)}$

Theorem 2.3. *Fix a positive integer ℓ. The Catalan functions $\{\mathfrak{s}_\mu^{(k)}\}_{k\geq 1, \mu\in \mathrm{Par}_\ell^k}$, satisfy the following properties:*

(horizontal dual Pieri rule) $\quad h_d^\perp \mathfrak{s}_\mu^{(k)} = \sum_{T\in \mathrm{SMT}^k_{(d)}(\mu)} t^{\mathrm{spin}(T)} \mathfrak{s}^{(k)}_{\mathrm{inside}(T)}$ *for all* $d \geq 0$; (2.5)

(vertical dual Pieri rule) $\quad e_d^\perp \mathfrak{s}_\mu^{(k)} = \sum_{T\in \mathrm{VSMT}^k_{(d)}(\mu)} t^{\mathrm{spin}(T)} \mathfrak{s}^{(k)}_{\mathrm{inside}(T)}$ *for all* $d \geq 0$; (2.6)

(shift invariance) $\quad \mathfrak{s}_\mu^{(k)} = e_\ell^\perp \mathfrak{s}^{(k+1)}_{\mu+1^\ell}$; (2.7)

(Schur function stability) $\quad$ *if* $k \geq |\mu|$, *then* $\mathfrak{s}_\mu^{(k)} = s_\mu$. (2.8)

$$s_\mu^{(k)}(x;t) = \sum_{\eta\in \mathbb{Z}_{\geq 0}^\infty,\ |\eta|=|\mu|} \ \sum_{T\in \mathrm{SMT}^k_\eta(\mu)} t^{\mathrm{spin}(T)} x^\eta$$

Thm 2.4: For $\mu \in \mathrm{Par}_\ell^k$, the k-Schur function $s_\mu^{(k)}$ is the Catalan function $\mathfrak{s}_\mu^{(k)}$

λ to be a partition, $|\lambda| = |\mu|$

Pf: $\langle \mathfrak{s}_\mu^{(k)}, h_\lambda\rangle = \langle h_\lambda^\perp \mathfrak{s}_\mu^{(k)}, 1\rangle$

$$h_\lambda = h_{\lambda_1} \cdot h_{\lambda_2} \cdots$$

$$\langle \cdots h_{\lambda_2}^\perp h_{\lambda_1}^\perp \mathfrak{s}_\mu^{(k)}, 1\rangle$$

$$= \left\langle \sum_{T\in \mathrm{SMT}^k_{(\lambda)}(\mu)} t^{\mathrm{spin}(T)} \mathfrak{s}^{(k)}_{\mathrm{inside}(T)}, 1 \right\rangle \quad (0,0,0,\ldots)$$

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m_λ, h_λ are dual bases,

$$\mathfrak{s}_\mu^k(x;t) = \sum_{\substack{\text{partitions } \lambda \\ |\lambda|=|\mu|}} \left(\sum_{T\in \mathrm{SMT}^k_{(\lambda)}(\mu)} t^{\mathrm{spin}(T)}\right) m_\lambda(x)$$

$$s_\mu^{(k)}(x;t) = \sum_{\eta\in \mathbb{Z}_{\geq 0}^\infty,\ |\eta|=|\mu|} \left(\sum_{T\in \mathrm{SMT}^k_\eta(\mu)} t^{\mathrm{spin}(T)}\right) x^\eta$$

$$\mathfrak{s}_\mu^k = s_\mu^k \qquad \square$$

Corollary 2.5.

(1) *The k-Schur functions* $s^{(k)}_\mu(\mathbf{x};t)$ *defined by* (2.9) *are symmetric functions.*

(2) *The k-Schur functions* $s^{(k)}_\mu(\mathbf{x};t)$ *are equal to Catalan functions defined by Chen-Haiman via a different root ideal than our* $\Delta^k(\mu)$ *(see Section 10).*

(3) *For* $\mu \in \mathrm{Par}^k_\ell$, $s^{(k)}_\mu(\mathbf{x};t)$ *is the* GL_ℓ*-equivariant Euler characteristic of a vector bundle on the flag variety determined by* μ *and* k.

(4) *Homology Schubert classes of* $\mathrm{Gr}_{SL_{k+1}}$ *are equal to the ungraded* ($t=1$) *version of this Euler characteristic.*

Thm 2.6' (k-Schur Branching Rule)

$$s^{(k)}_\mu = \sum_{T \in VSMT^{(k+1)}_{(\ell)}(\mu + 1^\ell)} t^{spin(T)} s^{(k+1)}_{inside(T)}$$

Theorem 2.3. *Fix a positive integer* ℓ. *The Catalan functions* $\{\mathfrak{s}^{(k)}_\mu\}_{k\ge 1,\mu\in\mathrm{Par}^k_\ell}$, *satisfy the following properties:*

(horizontal dual Pieri rule) $$h^\perp_d \mathfrak{s}^{(k)}_\mu = \sum_{T\in \mathrm{SMT}^k_{(d)}(\mu)} t^{\mathrm{spin}(T)} \mathfrak{s}^{(k)}_{\mathrm{inside}(T)} \quad \text{for all } d \ge 0; \quad (2.5)$$

(vertical dual Pieri rule) $$e^\perp_d \mathfrak{s}^{(k)}_\mu = \sum_{T\in \mathrm{VSMT}^k_{(d)}(\mu)} t^{\mathrm{spin}(T)} \mathfrak{s}^{(k)}_{\mathrm{inside}(T)} \quad \text{for all } d \ge 0; \quad (2.6)$$

(shift invariance) $$\mathfrak{s}^{(k)}_\mu = e^\perp_\ell \mathfrak{s}^{(k+1)}_{\mu+1^\ell}; \quad (2.7)$$

(Schur function stability) $$\text{if } k \ge |\mu|, \text{ then } \mathfrak{s}^{(k)}_\mu = s_\mu. \quad (2.8)$$

Thm (Schur Expansion): Let $\mu \in Par_\ell^{(k)}$, set $m = \max(|\mu|-k, 0)$, Then, the Schur expansion:

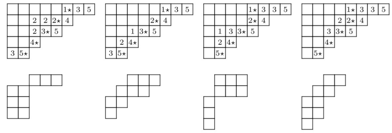
$$s^{(k)}_\mu = \sum_{T \in VSMT^{k+m}_{(\ell^m)}(\mu + m^\ell)} t^{spin(T)} s_{inside(T)}$$

Example

Example 2.13. According to Theorem 2.6, the expansion of $s^{(3)}_{22221}$ into 4-Schur functions is obtained by summing $t^{\mathrm{spin}(T)} s^{(4)}_{\mathrm{inside}(T)}$ over the set $\mathrm{VSMT}^4_{(5)}(33332)$ of vertical strong marked tableaux given below. Note that $86532 = \mathfrak{p}^{-1}(33332)$ is the outer shape of each diagram on the first line.

T

k-skew(inside(T))



inside(T)	3222	3321	33111	22221
spin(T)	2	2	2	0

$$s^{(3)}_{22221} = t^2 s^{(4)}_{3222} + t^2 s^{(4)}_{3321} + t^2 s^{(4)}_{33111} + s^{(4)}_{22221}.$$