Informal seminar on combinatorics and repr. theory

References:

[BMP] Blasiak, Morse, Pun
Demazure crystals and the Schur positivity of Catalan functions
arXiv: 2007.04952

[BMPS1] Blasiak, Morse, Pun, Summers
k-Schur expansions of Catalan fcts
arXiv: 1811.02490

[BMPS2] Blasiak, Morse, Pun, Summers
Catalan functions and k-Schur positivity
arXiv: 1804.03701

[B] Blasiak
The DARK side of generalized Demazure crystals
arXiv: 2007.04888

Ring of symmetric functions

A polynomial $p(x_1, \dots, x_n)$ is symmetric if

$$p(x_1, \dots, x_n) = p(x_{\omega(1)}, \dots, x_{\omega(n)}) \quad \forall \omega \in S_n$$

Homogeneous symmetric functions

$$h_d = \sum_{i_1 \leq \dots \leq i_d} x_{i_1} \cdots x_{i_d}$$

Ring of symmetric functions

$$\Lambda = \mathbb{Q}[h_1, h_2, \dots]$$

Schur functions

Schur functions $s_\lambda(x_1, \ldots, x_n)$ indexed by partition λ form a basis for Λ

$$s_\gamma = \det\left(h_{\gamma_i + j - i}\right)_{1 \le i, j \le \ell} \in \Lambda$$

$\gamma \in \mathbb{Z}^\ell$

Structure coefficients:
Littlewood-Richardson coefficients

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

$\in \mathbb{Z}_{\geq 0}$

Geometry: Grassmannian varieties

$G(k,n) = \{k\text{-dim subspaces of } \mathbb{C}^n\}$

Schubert varieties indexed by partitions λ

Cohomology ring: $H^*(G(k,n))$ with basis given by Schubert classes X_λ

As rings

$$H^*(G(k,n)) \cong \Lambda_n / \langle s_\lambda \mid \lambda \notin \underbrace{\boxed{\ }}_{k}{}^{n} \rangle$$

symmetric polyn. in n variables

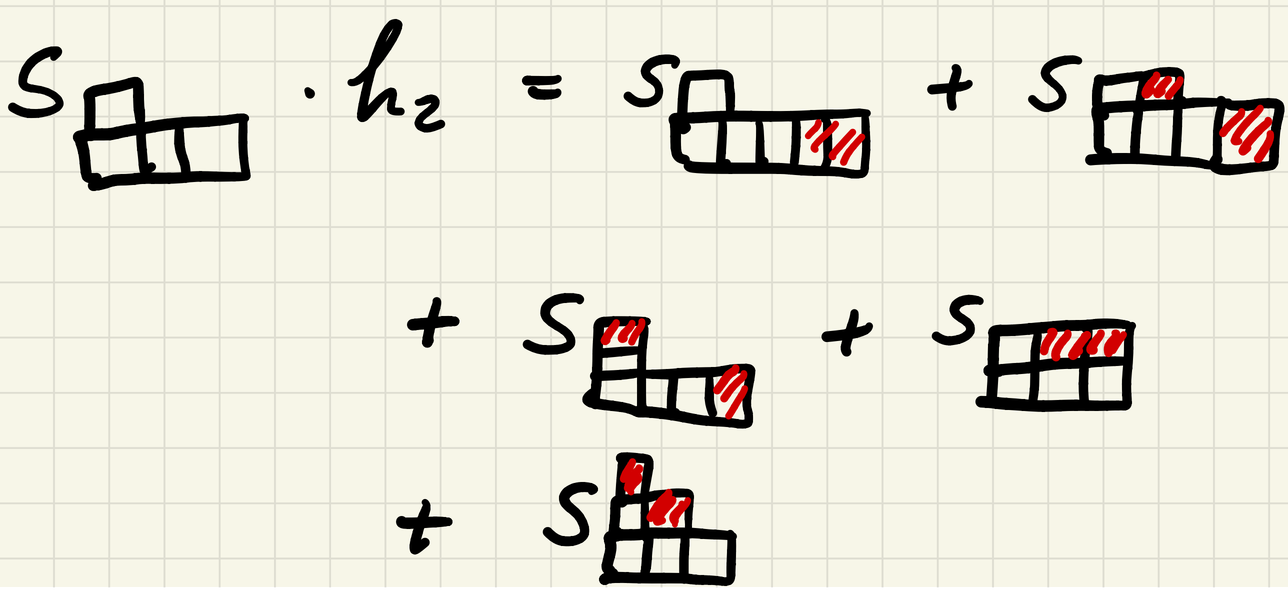
Combinatorics:

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^{wt(T)}$$

Pieri formula:

$$s_\lambda h_r = \sum_{\substack{\nu \\ \nu/\lambda \text{ horizontal} \\ r\text{-strip}}} s_\nu$$

<u>example</u>



k - Schur functions

$\Lambda_{(k)} = \mathbb{Q}[h_1, \ldots, h_k] \subset \Lambda$ subring

Pieri rule:

$$s_\lambda^{(k)} \cdot h_r = \sum_{\nu} s_\nu^{(k)}$$

ν k-bounded partition
ν/λ horizontal r-strip
$\nu^{(k)}/\lambda^{(k)}$ vertical r-strip

k-conjugate

Combinatorics:

$$s_\lambda^{(k)} = \sum_{\substack{T \text{ strong } k\text{-tableaux} \\ \text{of shape } \lambda}} x^{wt(T)}$$

Geometry:

$s_\lambda^{(k)}$ λ k-bounded partition
$(\lambda_1 \leq k)$

form a basis for $\Lambda_{(k)}$

Homology ring of affine Grassmannian

$$Gr = SL_{k+1}(\mathbb{C}((t)))/SL_{k+1}(\mathbb{C}[[t]])$$

$$H_*(Gr) \cong \Lambda_{(k)}$$

Structure coefficients:

$$s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\substack{\nu \\ k\text{-bounded}}} c_{\lambda\mu}^{\nu,k} s_\nu^{(k)}$$

related to 3-point Gromov-Witten invariants or quantum (co)homology

Catalan functions

Chen, Haiman
Panyushev

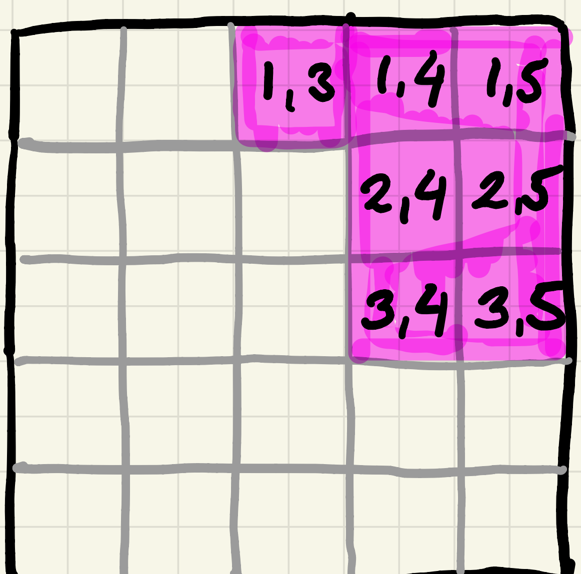
Def root ideal

A root ideal is an upper order ideal of the poset

$$\Delta_\ell^+ = \{(i,j) \mid 1 \le i < j \le \ell\}$$

Partial order: $(a,b) \le (c,d)$ if $a \geq c$ and $b \le d$

Example:



$$\psi = \{(1,3), (1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$$

is root ideal

<u>Def</u> root ideal $\Psi \subseteq \Delta_\ell^+$
$\gamma \in \mathbb{Z}^\ell$

The Catalan function is

$$H(\Psi;\gamma)(x;t) = \prod_{(i,j)\in\Psi} (1-tR_{ij})^{-1} s_\gamma(x)$$

Raising operator

$$R_{ij}\, s_\gamma = s_{\gamma+\epsilon_i-\epsilon_j}$$

Def k-Schur root ideal

μ k-bounded composition

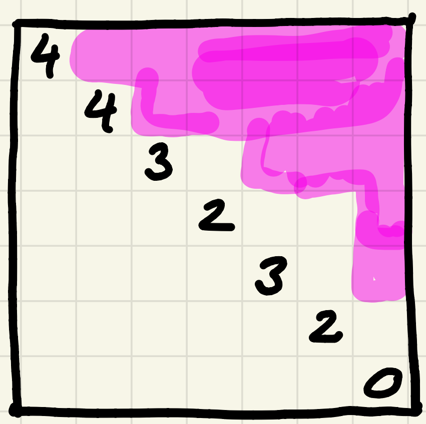
$$\Delta^k(\mu) = \{(i,j) \in \Delta_\ell^+ \mid k-\mu_i+i < j\}$$

$$\mathfrak{s}_\mu^{(k)} = H(\Delta^k(\mu);\mu)$$

$$= \prod_{i=1}^{\ell} \prod_{j=k+1-\mu_i+i}^{\ell} (1-tR_{ij})^{-1} s_\mu$$

example

$\mathfrak{s}_{4432320}$



Results [BMPS2]

Chen-Haiman conjecture

$$\mathfrak{s}_\lambda^{(k)}(x;t) = s_\lambda^{(k)}(x;t)$$

k-Schur branching

$$s_\lambda^{(k+1)} = \sum_\nu \underset{\mathbb{N}[t]}{\underset{\cap}{a_\lambda^\nu(t)}} \, s_\nu^{(k)}$$

Schur positivity

$$s_\lambda^{(k)}(x;t) = \sum_\mu \underset{\mathbb{N}[t]}{\underset{\cap}{a_\lambda^\mu(t)}} \, s_\mu$$

Results [BMP]

Conjecture Chen, Haiman

$$H(\Psi;\mu)(x;q) = \sum_{\lambda} K^{\Psi}_{\lambda\mu}(q)\, s_\lambda$$

$K^{\Psi}_{\lambda\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$

Parabolic subset $\Delta(\eta) \subset \Delta^+$

understood via Demazure crystals (Shimozono), KR crystals (Shimozono, Warnaar, Schilling), rigged configurations (Kirillov, Shimozono, Schilling)

Tame nonsymmetric Catalan fcts

$H(\Psi;\mu;w)$

w permutation

Tame nonsymmetric Catalan fcts are characters of $U_q(\widehat{sl_\omega})$-gener. Demazure crystals

→ DARK crystals

→ katabolism formulas

<u>Def</u> Nonsymmetric Catalan fcts

$$H(\Psi, \gamma, w)(x; q) = \pi_w \left(\mathrm{poly}\left(\prod_{(i,j) \in \Psi} \left(1 - q \frac{x_i}{x_j}\right)^{-1} x^\gamma \right) \right)$$

Demazure operator

polynomial truncation

Program

1. Catalan fcts and Hall-Littlewood polynomials Eugene
2. Recurrence for Catalan fcts Monica
3. Catalan fcts and k-Schur fcts Daniel
4. Nonsymmetric Catalan fcts, recursions, operator formula Wencin
5. Demazure crystals Nicolle
6. DARK crystals Jianping
7. Schur positivity for Catalan fcts Joseph
8. Connections to nonsymmetric Macdonald polynomials