Bijecting hidden symmetries

$\boxed{\text{arXiv: 2103.09551}}$

Goal: bijective proofs for

① Schur functions $s_{\delta_n/\mu} = s_{\delta_n/\mu'}$ [Stembridge 2004]

② Schur & Schur P-functions

$$s_{\delta_n/\rho^{lim}} = P_{\epsilon_n - \tau^{lim}}$$ [DeWitt 2012]

③ K-theoretic extensions

Schur Functions $\quad s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$

$SSYT(\lambda)$: semistandard Young tableaux of shape λ

$\cup$

$SYT(\lambda)$: standard Young tableau of shape λ

$\Downarrow$

S^λ : superstandard tableau $\quad$ E.g. $\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{matrix}$

Descent set for $T \in SYT(\lambda)$ $\quad \lambda \vdash n$

$Des(T) := \{ i \in [n-1] : i$ is strictly above $i+1 \}$

E.g. $T = \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \end{matrix}$ $\quad Des(T) = \{1, 4\}$

Reading word of T : read rows left to right, bottom to top

Eg. $r(T) = 25134$

Schur P- funtions $\mu = (\mu_1 > \mu_2 > \cdots > \mu_\ell > 0)$ a strict partion

$$P_\mu = \sum_{T \in ShSSYT'(\mu)} x^T$$

Marked shifted semistandard Young tableaux $ShSSYT'(\mu)$

- μ_i boxes at row i, starting at column i
- row weakly increasing, column strictly increasing
- $1' < 1 < 2' < 2 < \cdots < n' < n$
- no prime along diagonal
- at most one i each column
- at most one i' each row

E.g. $T = \begin{matrix} 1 & 1 & 2 & 3' & 4 \\ & 4 & 5 & 5 & \\ & & 6 & 9 & \end{matrix} \in ShSSYT'((5,3,2))$

Marked shifted standard Young tableaux $ShSYT'(\mu)$

- are $ShSSYT'(\mu)$
- $\mu \vdash n$, either i or i' show up $\forall\ 1 \le i \le n$

Descent set for $T \in ShSYT'(\mu)$ $\mu \vdash n$.

$Des(T) := \{ i \in [n-1] : i, (i+1)' \in T,$ OR i is strictly above $i+1$,

OR i' is strictly to the left of $(i+1)'\ \}$

E.g.

$$T = \begin{array}{cccc} 1 & 2 & 4' & 6' \\ & 3 & 5' & 8 \\ & & 7 & \end{array} \qquad Des(T) = \{2, 3, 5\}$$

$ShSYT'(\lambda) \xrightarrow{\ \circ\ } ShSYT'(\lambda)$ mark unmarked, unmark marked

E.g.

$$T^{\circ} = \begin{array}{cccc} 1 & 2' & 4 & 6 \\ & 3 & 5 & 8' \\ & & 7 & \end{array} \qquad Des(T^{\circ}) = \{1, 4, 6, 7\}$$

Lemma $\lambda \vdash n$, $T \in ShSYT'(\lambda)$, then

$$Des(T^{\circ}) = [n-1] \setminus Des(T)$$

Semistandard $\longleftrightarrow$ Standard $\times$ I_{Des}

Def $S \subseteq [n-1]$, $I_S := \{ (i_1 \le i_2 \le \cdots i_n) : k \in S \Rightarrow i_k < i_{k+1} \}$

$T \in SYT(\lambda)$ or $ShSYT'(\lambda)$, $\underline{i} = (i_1, \cdots, i_n)$

$\Rightarrow$ $T(\underline{i}) \in SSYT(\lambda)$ or $ShSSYT'(\lambda)$ by

- replace k w/ i_k (k' w/ i_k')

Thm $\lambda \vdash n$, μ strict partition of n.
Then $(T, \underline{i}) \mapsto T(\underline{i})$ is a bijection from

(a) $\displaystyle\bigcup_{T \in SYT(\lambda)} \{T\} \times I_{Des(T)} \longrightarrow SSYT(\lambda)$

(b) $\displaystyle\bigcup_{T \in ShSYT'(\mu)} \{T\} \times I_{Des(T)} \longrightarrow ShSSYT'(\mu)$

Cor $\lambda \vdash n$, μ strict partition of n.

$$S_\lambda = \sum_{T \in SYT(\lambda)} \sum_{\underline{i} \in I_{Des(T)}} x^{T(\underline{i})}, \qquad P_\mu = \sum_{T \in ShSYT'(\mu)} \sum_{\underline{i} \in I_{Des(T)}} x^{T(\underline{i})}$$

E.g.

$$S_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2 x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

① ② ③ ④ ⑤ ⑥ ⑦ ⑧

$$SYT((2,1)) = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} , \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right\}$$

$Des = \{2\}$ $\qquad$ $Des\ \{1\}$

① $1 \le 1 < 2$

② $1 \le 1 < 3$

④ $1 \le 2 < 3$

⑦ $2 \le 2 < 3$

③ $1 < 2 \le 2$

⑤ $1 < 2 \le 3$

⑥ $1 < 3 \le 3$

⑧ $2 < 3 \le 3$

Worley – Sagan insertion

$$\text{words} \longrightarrow \bigsqcup_{\lambda} \mathrm{ShSSYT}(\lambda) \times \mathrm{ShSYT}'(\lambda)$$

$$w \longmapsto (P_{SW}, Q_{SW})$$

weak insertion $x \to (r_1, \ldots, r_n)$, i is minimal s.t. $r_i > x$ or $n+1$ if $r_n \le x$. If $i \ne n+1$, x replace r_i, bump r_i.

strict insertion $x \to (r_1, \ldots, r_n)$, i is minimal s.t. $r_i \geq x$ or $n+1$ if $r_n \le x$. If $i \ne n+1$, x replace r_i, bump r_i.

Worley – Sagan bumping $x^i \to T$, start at $i=1$

- weak insert to row i
- if x^i bumps x^{i+1}

(a) x^{i+1} is first in its row, switch to column strict insertion

(b) x^{i+1} is NOT first in its row, weak insert to row $i+1$

E.g. $\underline{a} = (1, 7, \underline{5}, \underline{9}, 8, 3, 6, \underline{7, 2, 4, 3})$ ⑦

P_{sw}: 1 $\quad 1\,7$ $\quad \begin{array}{l} 1\,5 \\ \;\;7 \end{array}$ $\quad \begin{array}{l} 1\,5\,9 \\ \;\;7 \end{array}$ $\quad \begin{array}{l} 1\,5\,8 \\ \;\;7\,9 \end{array}$ $\quad \begin{array}{l} 1\,3\,7\,8 \\ \;\;\boxed{5}\,9 \end{array}$ $\quad \begin{array}{l} 1\,3\,6\,8 \\ \;\;5\,7 \\ \;\;\;\;9 \end{array}$

Q_{sw}: 1 $\quad 1\,2$ $\quad \begin{array}{l} 1\,2 \\ \;\;3 \end{array}$ $\quad \begin{array}{l} 1\,2\,4 \\ \;\;3 \end{array}$ $\quad \begin{array}{l} 1\,2\,4 \\ \;\;3\,5 \end{array}$ $\quad \begin{array}{l} 1\,2\,4\,6' \\ \;\;3\,5 \end{array}$ $\quad \begin{array}{l} 1\,2\,4\,6' \\ \;\;3\,5 \\ \;\;\;\;7 \end{array}$

P_{sw}: $\begin{array}{l} 1\,3\,6\,7 \\ \;\;\boxed{5}\,7\,8 \\ \;\;\;\;9 \end{array}$ $\quad \begin{array}{l} 1\,2\,5\,6\,7 \\ \;\;3\,7\,8 \\ \;\;\;\;9 \end{array}$ $\quad \begin{array}{l} 1\,2\,4\,6\,7 \\ \;\;3\,5\,8 \\ \;\;\;\;7\,9 \end{array}$ $\quad \begin{array}{l} 1\,2\,3\,6\,7 \\ \;\;3\,4\,7\,8 \\ \;\;\;\;5\,9 \end{array}$

Q_{sw}: $\begin{array}{l} 1\,2\,4\,6' \\ \;\;3\,5\,8 \\ \;\;\;\;7 \end{array}$ $\quad \begin{array}{l} 1\,2\,4\,6'\,9' \\ \;\;3\,5\,8 \\ \;\;\;\;7 \end{array}$ $\quad \begin{array}{l} 1\,2\,4\,6'\,9' \\ \;\;3\,5\,8 \\ \;\;\;\;7\,10' \end{array}$ $\quad \begin{array}{l} 1\,2\,4\,6'\,9' \\ \;\;3\,5\,8\,11' \\ \;\;\;\;7\,10' \end{array}$

Plactic relations

Knuth moves $acb \sim cab$, if $a \le b < c$ Knuth classes

$\underline{a} \equiv \underline{b}$ $bac \sim bca$, if $a < b \le c$

Shifted Knuth moves Knuth moves, and shifted Knuth classes

$\underline{a} \doteq \underline{b}$ exchange the first two entries

<u>Note</u> $\underline{a} \equiv \underline{b} \Longrightarrow \underline{a} \doteq \underline{b}$

Descent of a word $\mathrm{Des}(\underline{a}) := \{ i \in [n-1] : a_i > a_{i+1} \}$

$\underline{a} = (a_1, a_2, \cdots, a_p)$

THM (1) Worley–Sagan insertion is a bijection from words to $(P_{sw}, Q_{sw}) \in \mathrm{ShSSYT}(\lambda) \times \mathrm{ShSYT}'(\lambda)$, λ some shifted shape.

(2) $\underline{a} \doteq \underline{b}$ iff $P_{sw}(\underline{a}) = P_{sw}(\underline{b})$.

(3) $\mathrm{Des}(\underline{a}) = \mathrm{Des}(Q_{sw}(\underline{a}))$

[Sagan 1987
Worley 1984]

Thm $(-\underline{a}) = (-a_1, \cdots, -a_p)$. $Q_{sw}(-\underline{a}) = Q_{sw}(\underline{a})^{\circ}$