Uncrowding algorithm for hook-valued tableaux

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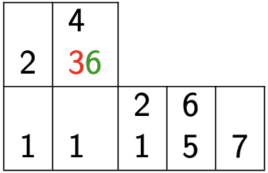
based on joint work with Joseph Pappe, Wencin Poh and Anne Schilling (preprint arXiv:2012.14975)



Informal seminar on combinatorics and representation theory
UC Davis
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1. Background and definitions

2. Uncrowding on HVT

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4. Applications

Stable symmetric Grothendieck polynomials

$$G_\lambda(\mathbf{x};\beta) = \sum_{T\in \mathrm{SVT}(\lambda)} \beta^{\mathrm{ex}(T)} x_1^{\#\text{of 1's}} x_2^{\#\text{of 2's}} \dots \qquad \text{(Buch 2002)}$$

SVT(λ) $=$ set of semistandard set-valued tableaux of shape λ

Excess in T is ex(T)

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Fill boxes of skew shape ν/λ with nonempty sets. Semistandardness:

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Example (Which one is a valid filling?)

$$\begin{array}{|c|c|c|} \cline{1-2} 34 & 45 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & 25 \\ \cline{2-3} \end{array} \qquad \begin{array}{|c|c|c|} \cline{1-2} 34 & 35 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & 456 \\ \cline{2-3} \end{array} \qquad \begin{array}{|c|c|c|} \cline{1-2} 2 & 35 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 14 & 56 \\ \cline{2-3} \end{array}$$

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Background

Set-valued tableaux

- K-theory of the Grassmannian

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Duality

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- Hook-valued tableaux

Variations of stable Grothendieck polynomials and their combinatorics

- Stable symmetric Grothendieck functions $G_\lambda^{(\beta)}$ by Fomin Kirillov 1994, Buch 2002
  No arm, generating function of set-valued tableaux

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- Stable canonical Grothendieck functions $G_\lambda^{(\alpha,\beta)}$ by Yeliussizov, 2017
  Generating functions of hook-valued tableaux

Semistandard hook-valued tableaux

Hook U

$$\begin{array}{|c|ccc}
\cline{1-1}
\ell_p \\
\cline{1-1}
\vdots \\
\cline{1-1}
\ell_1 \\
\hline
x & \multicolumn{1}{|c|}{a_1} & \multicolumn{1}{c|}{\dots} & \multicolumn{1}{c|}{a_q} \\
\hline
\end{array}$$

$x < \ell_1 < \ell_2 \cdots < \ell_p$

$x \leqslant a_1 \leqslant \cdots \leqslant a_q$

Semistandard hook-valued tableaux

Hook U

$$\begin{array}{|c|c|c|c|}
\cline{1-1} \ell_p & \multicolumn{3}{c}{} \\
\cline{1-1} \vdots & \multicolumn{3}{c}{} \\
\cline{1-1} \ell_1 & \multicolumn{3}{c}{} \\
\hline x & a_1 & \dots & a_q \\
\hline
\end{array}$$

$x < \ell_1 < \ell_2 \cdots < \ell_p$

$x \leqslant a_1 \leqslant \cdots \leqslant a_q$

Terminology

- Hook entry: $\mathrm{H}(U) = x$
- Arm: $\mathrm{A}(U) = \{a_1, \dots, a_q\}$
- Leg: $\mathrm{L}(U) = \{\ell_1, \dots, \ell_p\}$
- Extended leg: $\mathrm{L}^+(U) = \{x, \ell_1, \dots \ell_p\}$

Semistandard hook-valued tableaux

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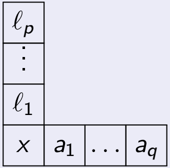

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Semistandard hook-valued tableau Yeliussizov 2017

Filling of Young diagram λ with hooks such that:

- $\max(A) \leqslant \min(B)$ whenever cell $\boxed{A}$ is left of $\boxed{B}$ in same row
- $\max(A) < \min(C)$ whenever cell $\boxed{A}$ is below $\boxed{C}$ in same column

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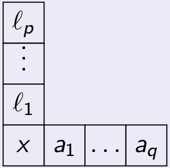

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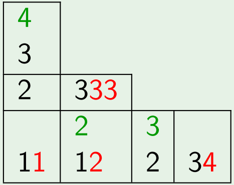
Example



Canonical stable Grothendieck polynomials

Generating set

$\mathrm{HVT}(\lambda) :=$ set of semistandard hook-valued tableaux of shape λ

For $H \in \mathrm{HVT}(\lambda)$, denote

- $a(H) =$ total number of cells in all arms
- $\ell(H) =$ total number of cells in all legs
- $\mathrm{wt}(H) = (\#\text{of } 1\text{'s}, \#\text{of } 2\text{'s}, \dots)$

Definition

$$G_\lambda^{(\alpha,\beta)}(\boldsymbol{x}) = \sum_{H \in \mathrm{HVT}(\lambda)} \alpha^{a(H)} \beta^{\ell(H)} \boldsymbol{x}^{\mathrm{wt}(H)}$$

Restricting letters to $1, 2, \dots, m$ is equivalent to restricting variables to $x_1, x_2, \dots, x_m$.

Crystal structure on hook-valued tableaux (Hawkes, Scrimshaw 2020)

Reading word

- Read tableau column-by-column, left to right
- Within each column:
  - read extended leg in each cell from top to bottom
  - read all remaining entries in weakly increasing order

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Example

$$T = \begin{array}{|l|l|}\hline \begin{matrix}4\\34\end{matrix} & \begin{matrix}5\\4\end{matrix} \\ \hline \begin{matrix}2\\11\end{matrix} & \begin{matrix}3\\233\end{matrix} \\ \hline \end{array}$$

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Each column reads

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i-pairing $(1 \leqslant i < m)$

- Assign $-$ to each i
- Assign $+$ to every $i+1$
- Successively pair each $+$ that is adjacent and to the left of a $-$
- Remove paired signs until nothing can be paired.

Example

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Example

- $i = 1$: 4321①4543233
- $i = 2$: 432114543233
- $i = 3$: 43211454323③

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- $i = 4: \; 43211\textcircled{4}543233$

Crystal structure on hook-valued tableaux (Hawkes, Scrimshaw 2020)

Crystal operator f_i

If there is no unpaired $-$, $f_i(T) = 0$. Otherwise, let cell B contain the rightmost unpaired i. Do one of the following in order:

(M) If $i+1$ is in $B^{\uparrow}$, remove an i from $\mathrm{A}(B)$ and add $i+1$ to $\mathrm{A}(B^{\uparrow})$.

(S) If i is in $B^{\rightarrow}$, remove the i from $\mathrm{L}^{+}(B^{\rightarrow})$ and add $i+1$ to $\mathrm{L}(B)$.

(N) Else, f_i changes the i in B into an $i+1$.

Example

$i = 1 : \ 4321\textcircled{1}4543233$ $\qquad$ $i = 2 : \ 432114543233$

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$i = 1:$ 4321$\textcircled{1}$4543233 $i = 2:$ 432114543233 $f_2(T) = 0$

Crystal structure on hook-valued tableaux (Hawkes, Scrimshaw 2020)

Crystal operator f_i

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(M) If $i + 1$ is in $B^{\uparrow}$, remove an i from $\mathrm{A}(B)$ and add $i + 1$ to $\mathrm{A}(B^{\uparrow})$.

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Example

$i = 1$: 4321①4543233 $\qquad$ $i = 2$: 432114543233 $\qquad$ $f_2(T) = 0$

$i = 3$: 43211454323③ $\qquad$ $i = 4$: 43211④543233

$$T = \begin{array}{|l|l|}\hline \begin{matrix}4\\34\end{matrix} & \begin{matrix}5\\4\end{matrix} \\ \hline \begin{matrix}2\\11\end{matrix} & \begin{matrix}3\\233\end{matrix} \\ \hline \end{array}, \qquad f_1(T) = \begin{array}{|l|l|}\hline \begin{matrix}4\\34\end{matrix} & \begin{matrix}5\\4\end{matrix} \\ \hline \begin{matrix}2\\12\end{matrix} & \begin{matrix}3\\233\end{matrix} \\ \hline \end{array},$$

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$i = 3:\ 43211454323\textcircled{3}$ $\qquad i = 4:\ 43211\textcircled{4}543233$

$$T = \begin{array}{|l|l|}\hline \begin{matrix}4\\34\end{matrix} & \begin{matrix}5\\4\end{matrix} \\ \hline \begin{matrix}2\\11\end{matrix} & \begin{matrix}3\\233\end{matrix} \\ \hline\end{array}, \quad f_1(T) = \begin{array}{|l|l|}\hline \begin{matrix}4\\34\end{matrix} & \begin{matrix}5\\4\end{matrix} \\ \hline \begin{matrix}2\\12\end{matrix} & \begin{matrix}3\\233\end{matrix} \\ \hline\end{array}, \quad f_3(T) = \begin{array}{|l|l|}\hline \begin{matrix}4\\34\end{matrix} & \begin{matrix}5\\44\end{matrix} \\ \hline \begin{matrix}2\\11\end{matrix} & \begin{matrix}3\\23\end{matrix} \\ \hline\end{array},$$

Crystal structure on hook-valued tableaux (Hawkes, Scrimshaw 2020)

Crystal operator f_i

If there is no unpaired $-$, $f_i(T) = 0$. Otherwise, let cell B contain the rightmost unpaired i. Do one of the following in order:

(M) If $i+1$ is in $B^{\uparrow}$, remove an i from $\mathrm{A}(B)$ and add $i+1$ to $\mathrm{A}(B^{\uparrow})$.

(S) If i is in $B^{\rightarrow}$, remove the i from $\mathrm{L}^{+}(B^{\rightarrow})$ and add $i+1$ to $\mathrm{L}(B)$.

(N) Else, f_i changes the i in B into an $i+1$.

Example

$i = 1:$ $4321\textcircled{1}4543233$ $\quad i = 2:$ 432114543233 $\quad f_2(T) = 0$

$i = 3:$ $43211454323\textcircled{3}$ $\quad i = 4:$ $43211\textcircled{4}543233$

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Uncrowding maps

Function	Combinatorial objects	Crystal structure	Uncrowding maps
$G_\lambda^{(\beta)}$	set-valued tableaux	Monical, Pechenik Scrimshaw 2018	$\sqcup_\mu \mathrm{SSYT}(\mu) \times \mathcal{F}(\mu/\lambda)$ Buch 2012
$G_\lambda^{(\alpha)}$	multiset-valued tableaux	Hawkes Scrimshaw 2019	$\sqcup_\mu \mathrm{SSYT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$ Hawkes, Scrimshaw 2019
$G_\lambda^{(\alpha,\beta)}$	hook-valued tableaux	Hawkes Scrimshaw 2019	?

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$G_\lambda^{(\alpha,\beta)}$	hook-valued tableaux	Hawkes Scrimshaw 2019	?

Why should we care about uncrowding maps?

- Bijections on generating sets of different functions.
- Yields symmetric function expansions.
- Gives crystal isomorphisms.

Uncrowding algorithm for SVT

Uncrowding operator Buch 2002; Bandlow, Morse 2012; Reiner, Tenner, Yong 2018; Chan, Pflueger 2019

- Identify topmost row in T containing a multicell.
- Let x be the largest letter in that row which lies in a multicell.
- Delete this x and perform RSK algorithm into the rows above.
- Repeat, resulting in a single-valued skew tableau.

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Example

4 3	5 4	
	2 1	5 2

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Example

$$\begin{array}{|c|c|c|}\hline \begin{matrix}4\\3\end{matrix} & \begin{matrix}{\color{red}5}\\4\end{matrix} & \\ \hline & \begin{matrix}2\\1\end{matrix} & \begin{matrix}5\\2\end{matrix} \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 5 & & \\ \hline \begin{matrix}{\color{red}4}\\3\end{matrix} & 4 & \\ \hline & \begin{matrix}2\\1\end{matrix} & \begin{matrix}5\\2\end{matrix} \\ \hline\end{array}$$

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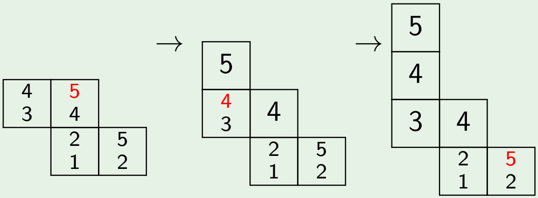

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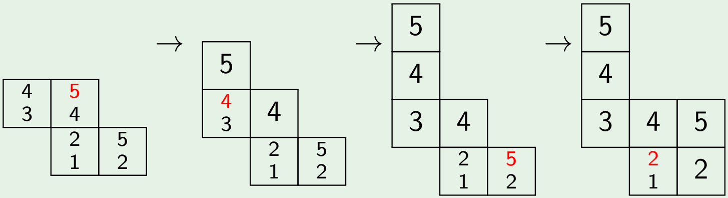

Uncrowding algorithm for SVT

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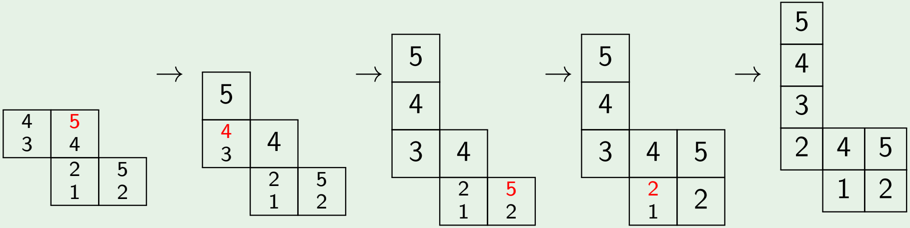

End of part I

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Uncrowding bumping $\mathcal{V}_b$ on $T \in \mathrm{HVT}$ (P., Pappe, Poh, Schilling, 2020)

If a cell contains nonzero arm, call it an armed cell.

1. Find the rightmost column c with an armed cell. Within column c, find the topmost armed cell (r, c). Denote the rightmost arm entry in cell (r, c) by a, and its largest leg entry by ℓ.
2. Look at $(c+1)$-st column and find the smallest number that is greater than or equal to a at cell $(\tilde{r}, c+1)$.
3. Move letter(s) over, maybe involves creating a new cell.

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Example

4 3	6 5		
2	4 333	5	
11	2 12	3 2	3

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2	4 333	5	
11	2 12	3 2	3

- Column 2

Uncrowding bumping $\mathcal{V}_b$ on $T \in$ HVT (P., Pappe, Poh, Schilling, 2020)

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11	2 12	3 2	3

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3. Move letter(s) over, maybe involves creating a new cell.

Example

4 3	6 5		
2	4 333	5	
11	2 12	3 2	3

- Column 2
- Row 2
- $a = 3$, $\ell = 4$
- $\tilde{r} = 1$

4 3	6 5		
2	4 33	5	
11	2 12	3 23	3

Uncrowding bumping $\mathcal{V}_b$ on $T \in$ HVT continued

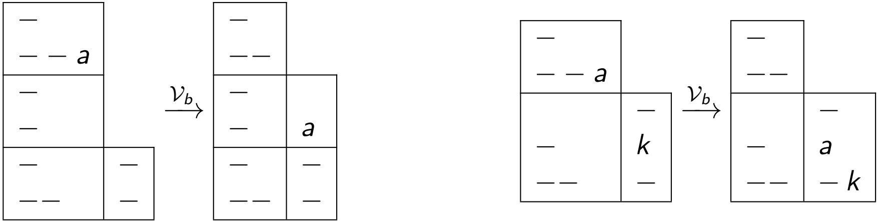


Figure: When $\tilde{r} \neq r$. Left: $(\tilde{r}, c+1)$ is a new cell; Right: $(\tilde{r}, c+1)$ is an existing cell.

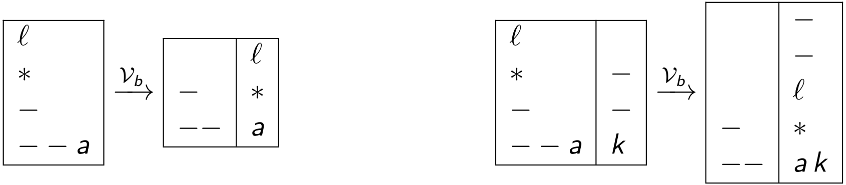


Figure: When $\tilde{r} = r$. Left: $(r, c+1)$ is a new cell; Right: $(r, c+1)$ is an existing cell.

Uncrowding map

Definition

Let $T \in \mathrm{HVT}(\lambda)$ with arm excess α. The uncrowding map

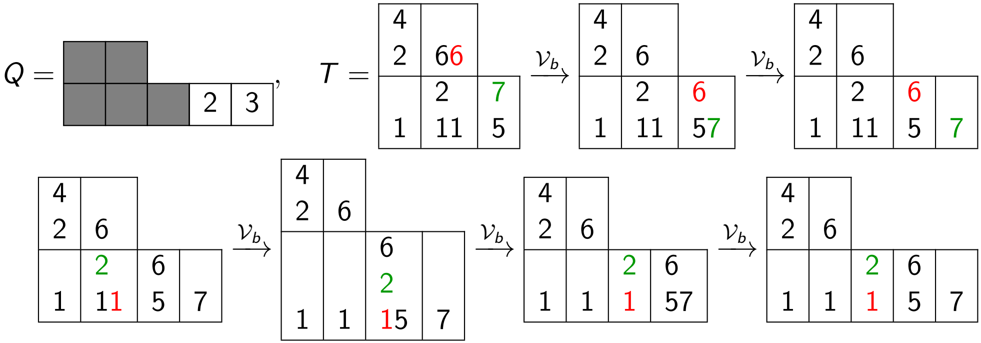
$$\mathcal{U}\colon \mathrm{HVT}(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} \mathrm{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$$

is defined by the following algorithm:

1. Let $P_0 = T$ and let Q_0 be the column-flagged increasing tableau of shape λ/λ.
2. For $1 \leqslant i \leqslant \alpha$, let $P_{i+1} = \mathcal{V}(P_i)$.
   - Let c be the index of the starting column
   - let $\tilde{c}$ be the column index of the cell $\mathrm{shape}(P_{i+1})/\mathrm{shape}(P_i)$
   - Fill Q_{i+1} with $\tilde{c} - c$ in the same new cell.

Define $\mathcal{U}(T) = (P(T), Q(T)) := (P_\alpha, Q_\alpha)$.

An example of uncrowding map



Properties

Theorem (P., Pappe, Poh, Schilling, 2020)

1. *If* $f_i(T) \neq 0$, *then* $f_i(P(T)) = P(f_i(T))$ *and* $Q(T) = Q(f_i(T))$.
2. *If* $e_i(T) \neq 0$, *then* $e_i(P(T)) = P(e_i(T))$ *and* $Q(T) = Q(e_i(T))$.

In other words, the diagram commutes:

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In other words, the diagram commutes:

$$\begin{tikzcd}
\mathrm{HVT} \arrow[r, "\mathcal{U}"] \arrow[d, "f_i"] & \mathrm{SVT} \times \hat{\mathcal{F}} \arrow[d, "f_i"'] \\
\mathrm{HVT} \arrow[r, "\mathcal{U}"] & \mathrm{SVT} \times \hat{\mathcal{F}}.
\end{tikzcd}$$

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Collorary 1

HVT^m is a type A_{m-1} Stembridge crystal.

Properties

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Collorary 3

$\mathcal{U}|_{\mathrm{MVT}}$ coincides with the uncrowding map on MVT described in Hawkes, Scrimshaw 2020 using RSK insertion.

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Crowding back

Crowding back

What can go wrong?

$$\mathcal{U}\colon \mathsf{HVT}(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)\,, \quad (S, Q) = \left(\begin{array}{cc} \boxed{3} & \\ \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{2} \end{array} , \begin{array}{cc} \blacksquare & \\ \blacksquare & \boxed{1} \end{array} \right).$$

Crowding back

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We say the cell $(1,2)$ in S practices social distancing.

Crowding back

What can go wrong?

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We say the cell $(1,2)$ in S practices social distancing.

Solution

- Restrict our domain to a subset of $\sqcup_{\mu\supseteq\lambda}\mathrm{SVT}(\mu)\times\hat{\mathcal{F}}(\mu/\lambda)$.
- $\mathrm{weight}(T_j^{(s)}) = \mathrm{weight}(S)$

Going back: one characterization! $\mathcal{C}_b$

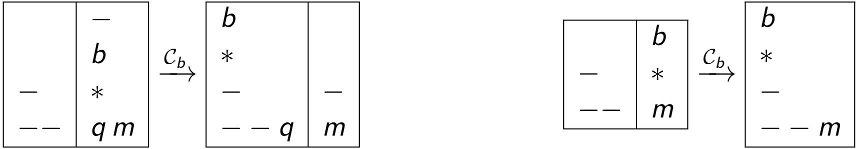


Figure: When $r' = r$. Left: (i) $\mathrm{A}_h(r, c) \neq \emptyset$. Right: (ii) $\mathrm{A}_h(r, c) = \emptyset$.

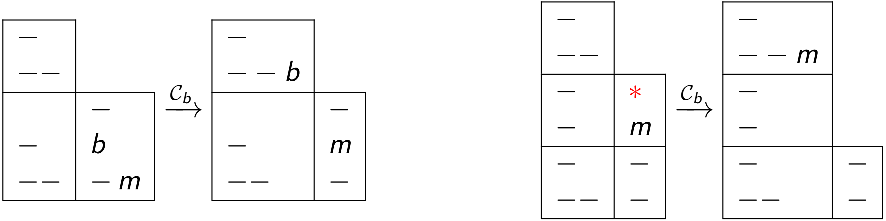
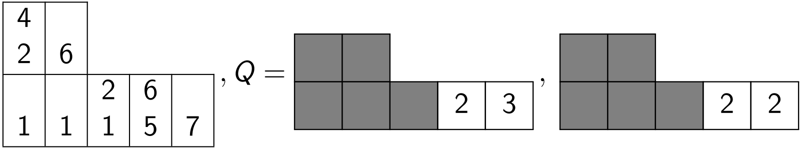


Figure: When $r' \neq r$. Left: $\mathrm{A}_h(r, c) \neq \emptyset$. Right: $\mathrm{A}_h(r, c) = \emptyset$.

An example on crowding map



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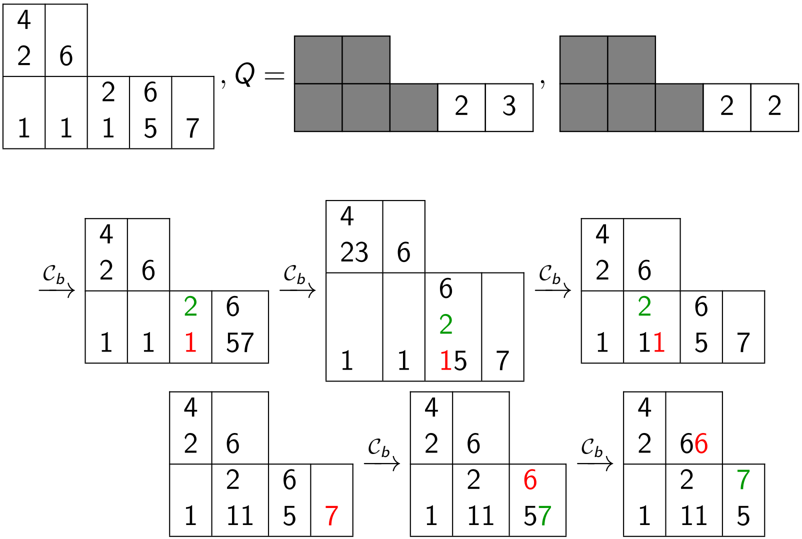

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Tableaux Schur expansion (TSE)

A symmetric function f_α is said to have a TSE if there is a set of (semistandard Young) tableaux $\mathbb{T}(\alpha)$ and a weight function $\mathrm{wt}_\alpha \colon \mathbb{T}(\alpha) \to R$ so that

$$f_\alpha = \sum_{T \in \mathbb{T}(\alpha)} \mathrm{wt}_\alpha(T) s_{\mathrm{shape}(T)}.$$

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$$f_\alpha = \sum_{T \in \mathbb{T}(\alpha)} \mathrm{wt}_\alpha(T) s_{\mathrm{shape}(T)}.$$

Theorem (Bandlow, Morse 2012)

Let f_α be a symmetric function with a TSE $f_\alpha = \sum_{T \in \mathbb{T}(\alpha)} \mathrm{wt}_\alpha(T) s_{\mathrm{shape}(T)}$ for some $\mathbb{T}(\alpha)$. Then we have

$$f_\alpha = \sum_{R \in \mathbb{R}(\alpha)} \mathrm{wt}_\alpha(R) G_{\mathrm{shape}(R)}(x; -1) = \sum_{S \in \mathbb{S}(\alpha)} \mathrm{wt}_\alpha(S) (-1)^{|S| - |\mathrm{shape}(S)|} g_{\mathrm{shape}(S)}(x; 1).$$

TSE for $G_\lambda^{(\alpha,\beta)}$

Proposition (P., Pappe, Poh, Schilling, 2020)

$$G_\lambda^{(\alpha,\beta)}(\boldsymbol{x}) = \sum_{T \in \mathbb{T}(\lambda)} \mathrm{wt}_\lambda(T) s_{\mathrm{shape}(T)}$$

$$\mathbb{T}(\lambda) = \{T \in \mathrm{SSYT}(\nu) \mid \nu \supseteq \lambda, T \text{ is of highest weight in the crystal graph}\}$$

$$\mathrm{wt}_\lambda(T) = \sum_{\mu : \lambda \subseteq \mu \subseteq \mathrm{shape}(T)} \alpha^{|\mu|-|\lambda|} \beta^{|\mathrm{shape}(T)|-|\mu|} \sum_{Q \in \mathcal{F}(\mathrm{shape}(T)/\mu)} \phi_\lambda(\mathcal{U}_{\mathrm{SVT}}^{-1}(T, Q))$$

$$\phi_\lambda(S) = |\{F \in \hat{\mathcal{F}} \mid (S, F) \in \mathrm{K}_\lambda\}|.$$

An example on $G^{(\alpha,\beta)}_{(2)}$

$G^{(\alpha,\beta)}_{(2)}(x) = s_2 + \beta s_{21} + 2\alpha s_3 + \cdots$. We have $\mathrm{wt}_{(2)}(T_1) = 1, \mathrm{wt}_{(2)}(T_2) = \beta, \mathrm{wt}_{(2)}(T_3) = 2\alpha$.

$$\mathbb{T}((2)) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \ldots \right\}.$$

$$T_1, \qquad T_2, \qquad T_3, \qquad \ldots$$

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$G_{(2)}^{(\alpha,\beta)}(x) = s_2 + \beta s_{21} + 2\alpha s_3 + \cdots$. We have $\mathrm{wt}_{(2)}(T_1) = 1, \mathrm{wt}_{(2)}(T_2) = \beta, \mathrm{wt}_{(2)}(T_3) = 2\alpha$.

$$\mathbb{T}((2)) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|cc}\cline{1-1} 2 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \ldots \right\}.$$

$$T_1, \qquad T_2, \qquad T_3, \qquad \ldots$$

$$\{S \in \mathbb{S}((2)) \mid P(\mathrm{word}(S)) = T_1\} = \{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \}$$

$$\{S \in \mathbb{S}((2)) \mid P(\mathrm{word}(S)) = T_2\} = \{ \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \begin{array}{|c|c|}\hline & 2 \\ 1 & 1 \\ \hline \end{array} \}.$$

An example on $G_{(2)}^{(\alpha,\beta)}$

$G_{(2)}^{(\alpha,\beta)}(x) = s_2 + \beta s_{21} + 2\alpha s_3 + \cdots$. We have $\mathrm{wt}_{(2)}(T_1) = 1, \mathrm{wt}_{(2)}(T_2) = \beta, \mathrm{wt}_{(2)}(T_3) = 2\alpha$.

$$\mathbb{T}((2)) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \ldots \right\}.$$

$$T_1, \qquad T_2, \qquad T_3, \qquad \ldots$$

$$\{S \in \mathbb{S}((2)) \mid P(\mathrm{word}(S)) = T_1\} = \{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \}$$

$$\{S \in \mathbb{S}((2)) \mid P(\mathrm{word}(S)) = T_2\} = \{ \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline & 2 \\ 1 & 1 \\ \hline \end{array} \}.$$

$$G_{(2)}^{(\alpha,\beta)}(x) = g_{(2)}(x;1) + \beta(g_{(21)}(x;1) - g_{(2)}(x;1)) + \cdots .$$

An example on $G^{(\alpha,\beta)}_{(2)}$ continued

$$\{R \in \mathbb{R}((2)) \mid P(\mathrm{word}(R)) = T_1\} = \{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array}, \ldots \}$$

$$\{R \in \mathbb{R}((2)) \mid P(\mathrm{word}(R)) = T_2\} = \{ \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array}, \ldots \}$$

An example on $G_{(2)}^{(\alpha,\beta)}$ continued

$$\{R \in \mathbb{R}((2)) \mid P(\mathrm{word}(R)) = T_1\} = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}, \ldots \right\}$$

$$\{R \in \mathbb{R}((2)) \mid P(\mathrm{word}(R)) = T_2\} = \left\{ \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}, \ldots \right\}$$

Applying the expansion formulas, we obtain

$$\begin{aligned} G_{(2)}(x;\alpha,\beta) = & (G_{(2)}(x;-1) + G_{(21)}(x;-1) + G_{(22)}(x;-1) + G_{(211)}(x;-1) + \cdots) \\ & + \beta(G_{(21)}(x;-1) + G_{(22)}(x;-1) + 2G_{(211)}(x;-1) + \cdots) + \cdots \end{aligned}$$

End of part II

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