Combinatorial descriptions of some representations of (degenerate) affine and double affine Hecke algebra of type C

Yue Zhao

University of California, Davis

April 26, 2021

Outline

- $\mathcal{Y}$-semisimple representations and intertwining operators
- Combinatorial descriptions in type A case
- Schur-Weyl type duality
- Invariant spaces
- A basis of invariant spaces
- $\mathcal{Y}$-actions
- Combinatorial descriptions in type C case

$\mathcal{Y}$-semisimplicity

Definition ($\mathcal{Y}$-eigenspaces)

Since $\mathcal{Y} = \mathbb{C}\{y_1, \cdots, y_n\}$ is a commutative subalgebra of $H_n = H(W, y)$, then we can consider $\mathcal{Y}$-eigenspaces. Let M be an H_n module, for each function $\zeta : \{1, 2, \ldots, n\} \to \mathbb{C}$, define the subspace of M

$$M_\zeta = \{v \in M | y_i v = \zeta(i) v \text{ for } i \in [1, n]\}.$$

In the quantum case, we have $Y_i^{\pm}$ instead of y_i.

Definition($\mathcal{Y}$-semisimple)

Consider the case we have the decomposition

$$M = \bigoplus_\zeta M_\zeta,$$

in this case we call M $\mathcal{Y}$-semisimple.

Each ζ such that $M_\zeta \neq 0$ is called a $\mathcal{Y}$-weight. M_ζ is the weight space of weight ζ and $v \in M_\zeta$ is a weight vector of weight ζ.

Intertwining operators

Definition

For each $i \in [0, n-1]$, define

$$\phi_i = [s_i, y_i],$$

and for γ_n, define

$$\phi_n = [\gamma_n, y_n].$$

Let $W \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, for each $w \in W$, it has a reduced expression $w = s_{i_1} s_{i_2} \ldots s_{i_m}$, $l(w) = m$. Here $s_n = \gamma_n$. Define

$$\phi_w = \phi_{i_1} \phi_{i_2} \ldots \phi_{i_m}.$$

This is well-defined since $\phi_1, \ldots, \phi_n$ satisfy the same braid relations of type C.

A basic fact about the intertwining operators

$$\phi_w M_\zeta \subset M_{w \cdot \zeta},$$

where $w \cdot \zeta = \zeta \circ w^{-1}$.

Schur-Weyl type duality

Jordan-Ma functor (Quantum case)

$$\mathcal{F}_n^{\sigma,\eta,\tau} : \{U_q(\mathfrak{gl}_N) - \text{modules}\} \longrightarrow \{AHA - \text{representations of type } C\}$$

$$\mathcal{F}_{n,\psi,\omega,\iota}^{\sigma,\eta,\tau} : \{D_U - \text{modules}\} \longrightarrow \{DAHA - \text{representations of type } C\}$$

Etingof-Freund-Ma functor (Degenerate case)

$$F_{n,p,\mu} : \{GL_N - modules\} \longrightarrow \{dAHA - \text{representations of type } C\}$$

$$F_{n,p,\mu}^{\lambda} : \{\mathcal{D} - modules\} \longrightarrow \{dDAHA - \text{representations of type } C\}$$

Invariant spaces of Etingof-Freund-Ma functor

Invariant spaces

For a finite dimensional irreducible module V^ξ with highest weight $\xi \in P^+$, the invariant space

$$F_{n,p,\mu}(V^\xi) = Hom_{\mathfrak{t}_0}(1_{\mu\chi}, V^\xi \otimes V^{\otimes n}),$$

and

$$F^\lambda_{n,p,\mu}(A(GL_N)) = Hom_{\mathfrak{t}_0 \boxtimes \mathfrak{t}_0}(1_{\lambda\chi} \boxtimes 1_{\mu\chi}, A(GL_N) \otimes V^{\otimes n})$$

where $\mu, \lambda \in \mathbb{C}$ and χ is a character of $\mathfrak{t}$.

Classical symmetric pair

Let $N = p + q$, we have a classical symmetric pair $(\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_q)$. Let $\mathfrak{t} = \mathfrak{gl}_p \times \mathfrak{gl}_q$ and $\mathfrak{t}_0 = \{A \in \mathfrak{gl}_p \times \mathfrak{gl}_q | tr(A) = 0\}$.

Character χ

Define a character χ on $t = \mathfrak{gl}_p \times \mathfrak{gl}_q$ by

$$\chi(\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}) = qtrX_1 - ptrX_2.$$

Invariant spaces of Jordan-Ma functor

Invariant spaces

For a finite dimensional irreducible module V^ξ with highest weight $\xi \in P^+$, the invariant space

$$\mathcal{F}_n^{\sigma,\eta,\tau}(V^\xi) = Hom_{B_\sigma}(1_{\chi_\tau^\eta}, V^\xi \otimes V^{\otimes n}),$$

and

$$\mathcal{F}_{n,\psi,\omega,\iota}^{\sigma,\eta,\tau}(A_q(GL_N)) = Hom_{B'_\psi \boxtimes B_\sigma}(1_{\lambda_\iota^\omega} \boxtimes 1_{\chi_\tau^\eta}, A_q(GL_N) \otimes_2 (1 \boxtimes V_1) \otimes_2 \cdots \otimes_2 (1 \boxtimes V_n)),$$

where $\eta, \tau, \sigma \in \mathbb{C}$, χ_τ^η is a character of B_σ and λ_ι^ω is a character of B'_ψ.

Coideal subalgebras

Here $B_\sigma \subset U_q(\mathfrak{gl}_N)$ is a left coideal subalgebra defined via l-operators and $B'_\psi \subset U_q(\mathfrak{gl}_N)$ is a right coideal subalgebra.

Quantum symmetric pair

Here $(U_q(\mathfrak{gl}_N, B_\sigma))$ is a quantum symmetric pair. The quasi classical limit of the quantum symmetric pair is the classical symmetric pair.

Tensor product $V^{\xi} \otimes V^{\otimes n}$

We have the decomposition as follows,

$$V^{\xi} \otimes V^{\otimes n} = \bigoplus_{(\nu^{(0)},\nu^{(1)},\cdots,\nu^{(n)})} V^{\nu^{(n)}},$$

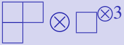
the direct sum runs through all the squences $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(n)})$ such that $\nu^{(0)} = \xi$, $\nu^{(i)} = \nu^{(i-1)} + \epsilon_{k_i}$ for some $k_i = 1, \cdots, N$ and each $\nu^{(i)} \in P^+$ for $i = 0, 1, \cdots, n$.
For any $\xi, \nu \in P^+$, we denote also by ν/ξ the subset of $\mathbb{Z}^2$ $\{(i, j_i) | i = 1, \cdots, N, \xi_i + 1 \leqslant j_i \leqslant \nu_i\}$.
For a sequence $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(n)})$, define a map $T_{(\nu^{(0)},\nu^{(1)},\cdots,\nu^{(n)})}$

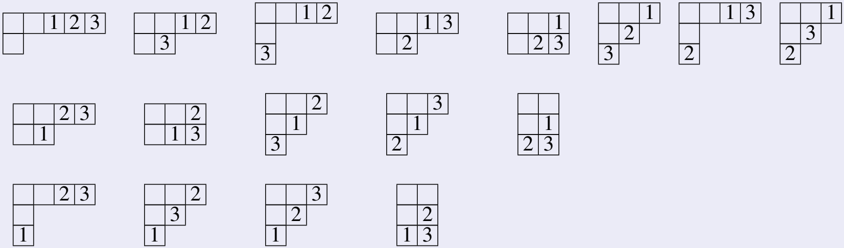
$$\nu^{(n)}/\xi \to \{1, 2, \cdots, n\}$$
$$(k_i, \nu^{(i)}_{k_i}) \mapsto i,$$

for $i = 1, \cdots, n$. Then $T_{(\nu^{(0)},\nu^{(1)},\cdots,\nu^{(n)})}$ is a standard tableau of shape $\nu^{(n)}/\xi$. So the tensor product $V^{\xi} \otimes V^{\otimes n}$ decomposes to a direct sum of irreducible $U_q(\mathfrak{gl}_N)$-modules indexed by sequences $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(n)})$ such that $\nu^{(0)} = \xi$ and $T_{(\nu^{(0)},\nu^{(1)},\cdots,\nu^{(n)})}$ is a standard tableau.

Decomposition of $V^{\xi} \otimes V^{\otimes n}$

By Pieri rule, $V^{\xi} \otimes V^{\otimes n}$ decomposes to a sum of irreducible GL_N modules which are indexed by skew tableaux and multiplicity free.

Example: 



$\nu^{(0)} = \xi = (2, 1, 0)$ $\quad$ $\nu^{(1)} = (3, 1, 0)$ $\quad$ $\nu^{(2)} = (3, 1, 1)$ $\quad$ $\nu^{(3)} = (3, 2, 1)$

Nonzero invariant spaces

Degenerate (Okada's Theorem)

For any two rectangular shapes (a^p) and (b^q), where a and b are nonnegative integers and $p \leqslant q$, then

$$s_{(a^p)} \times s_{(b^q)} = \sum c^{\nu}_{(a^p)(b^q)} s_{\nu},$$

where $c^{\nu}_{\eta\tau} = 1$ when ν satisfies the condition

$$\nu_i + \nu_{p+q-i+1} = a + b, \quad i = 1, \cdots, p \tag{1}$$
$$\nu_i = b, \quad i = p+1, \cdots, q \tag{2}$$

and $c^{\nu}_{\eta\tau} = 0$ otherwise.

Quantum case

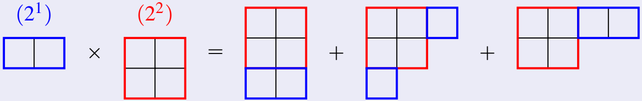
In the case $\sigma - \tau$ is an even number, the invariant space $Hom_{B_\sigma}(1_{\chi^{\eta}_{\tau}}, V^{\nu})$ is either 0 or a one-dimensional vector space. The invariant space is nonzero if and only if $\nu \in P^+$ and

$$\nu_i = \frac{\eta + \sigma - \tau}{2}, \quad i = p+1, \cdots, N-p, \tag{3}$$
$$\nu_i + \nu_{N-i+1} = \eta, \quad i = 1, \cdots, p. \tag{4}$$

Example

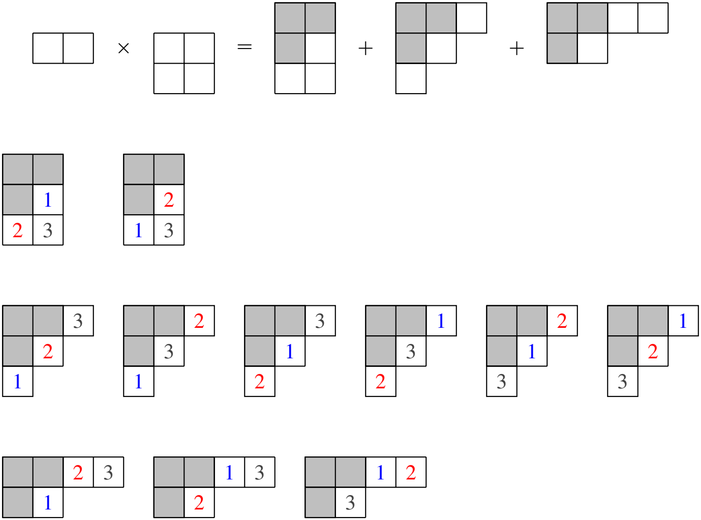
Let $M = V^{\yng(2,1)}$ be a GL_3-module, $n = 3$, $p = 1$ and $\mu = 0$.
Then $(a^p) = (2^1)$ and $(b^q) = (2^2)$.

By Okada's theorem, we could compute the shapes ν such that the invariant space is nonzero.



$$\begin{aligned}
&Hom_{\mathfrak{t}_0}(1_{\mu\chi}, V^{\xi} \otimes V^{\otimes 3}) \\
=&Hom_{GL_1 \times GL_2}(det^a \boxtimes det^b, V^{\xi} \otimes V^{\otimes 3}) \\
=&\bigoplus_{(\nu^{(0)},\cdots,\nu^{(3)})} Hom_{GL_1 \times GL_2}(det^a \boxtimes det^b, V^{\nu^{(3)}}).
\end{aligned}$$

A basis of the invariant space $F(V^{\boxplus})$



A basis of the invariant space

Degenerate case

The invariant space $Hom_{\mathfrak{t}_0}(1_{\mu\chi}, V^{\xi} \otimes V^{\otimes n})$ has a basis indexed by the collection of standard tableaux T of shape ν/ξ such that ν satisfies (1)-(2).

Quantum case

The invariant space $Hom_{B_\sigma}(1_{\chi^{\eta}_{\tau}}, V^{\xi} \otimes V^{\otimes n})$ has a basis indexed by the collection of standard tableaux T of shape ν/ξ such that ν satisfies (3)-(4).

$\mathcal{Y}$-actions

Quantum case

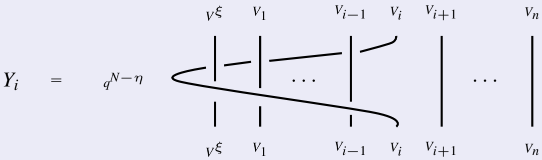


Figure: The action of Y_i

Degenerate case: generator y_i

The generator y_i acts by

$$-\Sigma_{i|j}(E_i^j)_0 \otimes (E_j^i)_k + \frac{p-q-\mu N}{2}\gamma_k + \frac{1}{2}\Sigma_{t>k}s_{kt} - \frac{1}{2}\Sigma_{t<k}s_{kt} + \frac{1}{2}\Sigma_{t\neq k}s_{kt}\gamma_k\gamma_t.$$

$\mathcal{Y}$-actions in terms of contents

Let v_T be a basis element corresponding to the standard tableau T.

Quantum case

Then the action of Y_i on v_T is multiplying by the scalar

$$q^{2cont_T(i)-\eta+N},$$

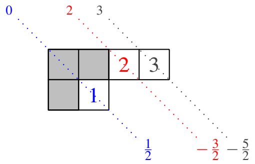
where $cont_T(i)$ is the content of the box filled by i.

Degenerate case

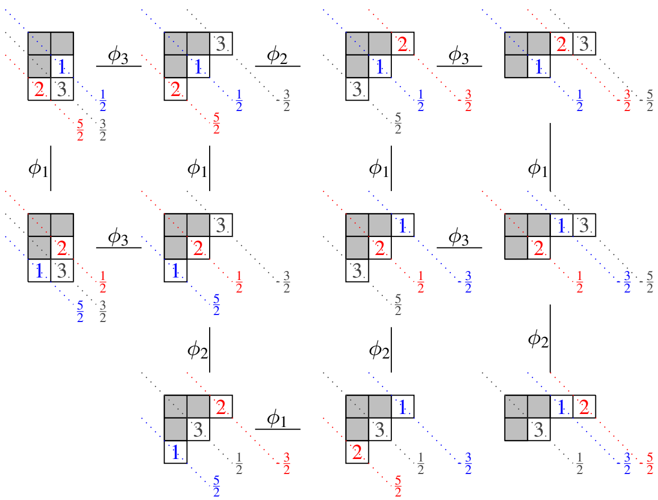
Then the action of y_i on v_T is multiplying by the scalar

$$-cont_T(i)+\frac{|\xi|+n}{N}-\frac{N}{2}-\frac{\mu(p-q)}{2},$$

where $cont_T(i)$ is the content of the box filled by i.



Intertwining operators and $\mathcal{Y}$ weight vectors



Invariant spaces: DAHA and dDAHA

Invariant spaces $\mathcal{F}(A_q(GL_N))$ and $F(A(GL_N))$

$$\mathcal{F}^{\sigma,\eta,\tau}_{n,\psi,\omega,\iota}(A_q(GL_N)) = Hom_{B'_\psi \boxtimes B_\sigma}(1_{\lambda^\omega_\iota} \boxtimes 1_{\chi^\eta_\tau}, (\bigoplus_{\beta \in P^+} V^{\beta^\vee} \boxtimes V^\beta) \otimes_2 (1 \boxtimes V_1) \otimes_2 \cdots \otimes_2 (1 \boxtimes V_n))$$

$$F^\lambda_{n,p,\mu}(A(GL_N)) = Hom_{\mathfrak{t}_0 \boxtimes \mathfrak{t}_0}(1_{\lambda\chi} \boxtimes 1_{\mu\chi}, (\bigoplus_{\beta \in P^+} V^{\beta^\vee} \boxtimes V^\beta) \otimes_2 (1 \boxtimes V_1) \otimes_2 \cdots \otimes_2 (1 \boxtimes V_n))$$

First, compute $\beta^\vee \in P^+$ such that

$$Hom_{\mathfrak{t}_0}(1_{\lambda\chi}, V^{\beta^\vee}) \neq 0.$$

For each $\beta^\vee \in P^+$ such that $Hom_{\mathfrak{t}_0}(1_{\lambda\chi}, V^{\beta^\vee}) \neq 0$, we compute

$$Hom_{\mathfrak{t}_0}(1_{\lambda\chi}, V^\beta \otimes V^{\otimes n}).$$

$F(A(GL_N))$

The invariant space $F^{\lambda}_{n,p,\mu}(A(GL_N))$ has a basis indexed by the collection of sequences $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(n)})$ such that $\nu^{(0)}$ satisfying

$$\nu_i^{(0)} = -t, i = p+1, \cdots, q, \tag{5}$$

$$\nu_i^{(0)} + \nu_{N-i+1}^{(0)} = -s-t, i = 1, \cdots, p. \tag{6}$$

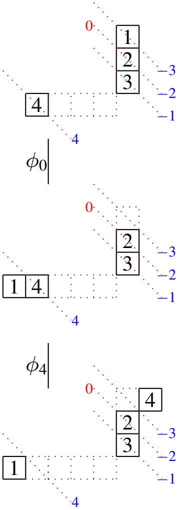
and $\nu^{(n)}$ satisfying

$$\nu_i^{(n)} = b, i = p+1, \cdots, q, \tag{7}$$

$$\nu_i^{(n)} + \nu_{N-i+1}^{(n)} = a+b, i = 1, \cdots, p. \tag{8}$$

Equivalently, a basis indexed by the collection of standard tableaux T of shape ν/β such that ν satisfying (7)-(8) and β satifying (5)-(6).

Consider the image of the coordinate ring of GL_4, with $n = 4$, $p = 1$, $\mu = -1$ and $\lambda = 1$ under the Etingof-Freund-Ma functor. The following figure gives some basis elements of the invariant space.



There is an isomorphism between H and $\hat{H}$

$$\begin{aligned}\sigma : & H \to \hat{H} \\ & s_i \mapsto \hat{s}_{n-i}, i = 1, \cdots, n-1 \\ & s_0 \mapsto \hat{s}_n, \\ & s_n \mapsto \hat{s}_0 \\ & y_i \mapsto \hat{y}_{n-i+1}, i = 1, \cdots, n.\end{aligned}$$

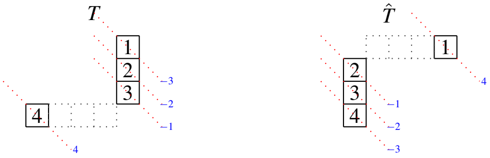
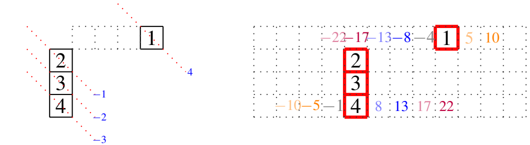
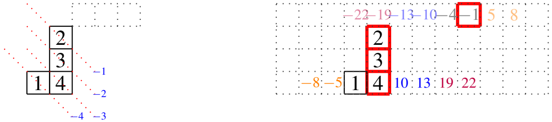


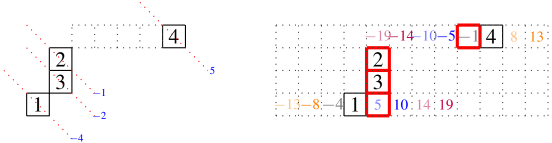
Figure: A standard tableau T and the corresponding standard tableau $\hat{T}$



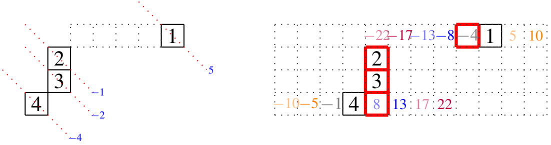
Applying ϕ_0, then we get



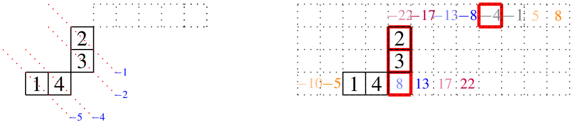
Applying ϕ_4, then we get



Applying $\phi_3\phi_2\phi_1\phi_2\phi_3$, then we get



Applying ϕ_0, then we get



Applying ϕ_4, then we get