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Stanley symmetric function

Definition S_n is the symmetric group generated by the simple transpositions $s_1, s_2, \ldots, s_{n-1}$ with relations

$$s_i^2 = 1 \qquad 1 \le i \le n-1$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad 1 \le i < n-1$$

$$s_i s_j = s_j s_i \qquad |i-j| > 1$$

$w \in S_n$: $i_1 \ldots i_\ell$ is a reduced word for w if $w = s_{i_1} \ldots s_{i_\ell}$ and ℓ is shortest with this property.

A word $a_1 a_2 \ldots a_\ell$ is decreasing if $a_1 > a_2 > \ldots > a_\ell$

$v_1 v_2 \dots v_\ell$ is a decreasing factorization of $w \in S_n$ if

- $w = v_1 v_2 \dots v_\ell$
- v_i is decreasing
- $\ell(w) = \ell(v_1) + \dots + \ell(v_\ell)$

Example

$w = s_1 s_3 s_2 s_3 \in S_4 \Rightarrow$ reduced words 1323, 3123, 1232

Decreasing factorization:

(1)(32)(3) (1)(3)(2)(3)

(31)(2)(3) (3)(1)(2)(3)

(1)(2)(32) (1)(2)(3)(2)

Definition The Stanley symmetric function is

$$F_w(x_1, x_2, \ldots) = \sum_{\substack{w = v_1 v_2 \ldots v_\ell \\ \text{decreasing fact.}}} x_1^{\ell(v_1)} \cdots x_\ell^{\ell(v_\ell)}$$

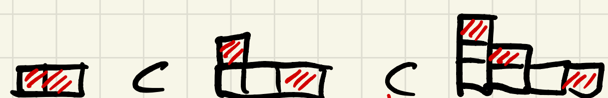
Example

$$F_{s_1 s_3 s_2 s_3} = m_{211} + 3\, m_{1111}$$

Theorem (Edelman-Greene) $F_w = \sum_\lambda a_{w\lambda}\, s_\lambda$, $a_{w\lambda} \in \mathbb{Z}_{\geq 0}$

Schur functions

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^{wt(T)}$$

$T = \emptyset \subset$  or $T = \begin{array}{|c|c|c|c|} \hline 3 & \multicolumn{3}{c}{} \\ \cline{1-2} 2 & 3 & \multicolumn{2}{c}{} \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array}$

horizontal strips

Affine Stanley symmetric functions

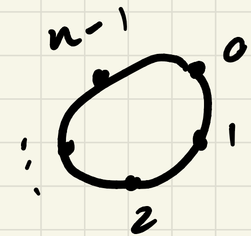
Definition Denote by $\tilde{S}_n$ the affine symmetric group generated by the simple reflections $s_0, s_1, \ldots, s_{n-1}$:

$$s_i^2 = 1 \qquad \forall i = 0, 1, \ldots, n$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_i s_j = s_j s_i \qquad |i-j| \geq 2$$

indices are taken modulo n



Note $S_n \subseteq \tilde{S}_n$ with generators $s_1, s_2, \ldots, s_{n-1}$

Definition A word $a_1 a_2 \ldots a_\ell$ with letters in $\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}$ is cyclically decreasing if

(1) each letter occurs at most once

(2) if i and $i+1$ occur, $i+1$ precedes i

Example $n=4$, $\mathbb{Z}/4\mathbb{Z} = \{0,1,2,3\}$

cyclically decreasing 321, 032, 31, 13

Definition $w \in \tilde{S}_n$

A *cyclically decreasing factorization* of $w \in \tilde{S}_n$ is a factorization $w = v_1 v_2 \cdots v_r$, where each v_i is cyclically decreasing and $\ell(w) = \ell(v_1) + \dots + \ell(v_r)$.

Definition *Affine Stanley symmetric function* for $w \in \tilde{S}_n$

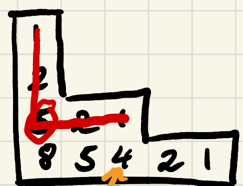
$$\tilde{F}_w = \sum_{\substack{w = v_1 \cdots v_r \\ \text{cycl. decreasing factorization}}} x_1^{\ell(v_1)} x_2^{\ell(v_2)} \cdots x_r^{\ell(v_r)}$$

Cores

Definition An n-core is a partition s.t. no n-ribbon can be removed.

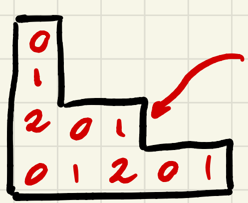
Or: An n-core is a partition s.t. no hook length is divisible by n.

Example



3 - core

hook length



residues

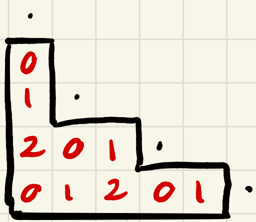
Action of $\tilde{S}_n$ on n-cores:

s_i adds all addable outer corners of n-core λ of residue i

or removes all removable corners of n-core λ of residue i

Example

$s_2 \cdot$



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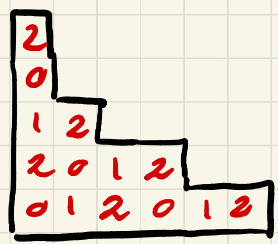
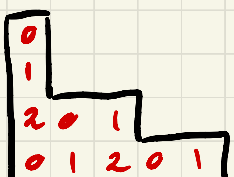


$s_1 \cdot$



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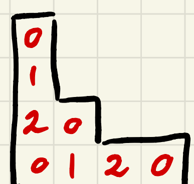
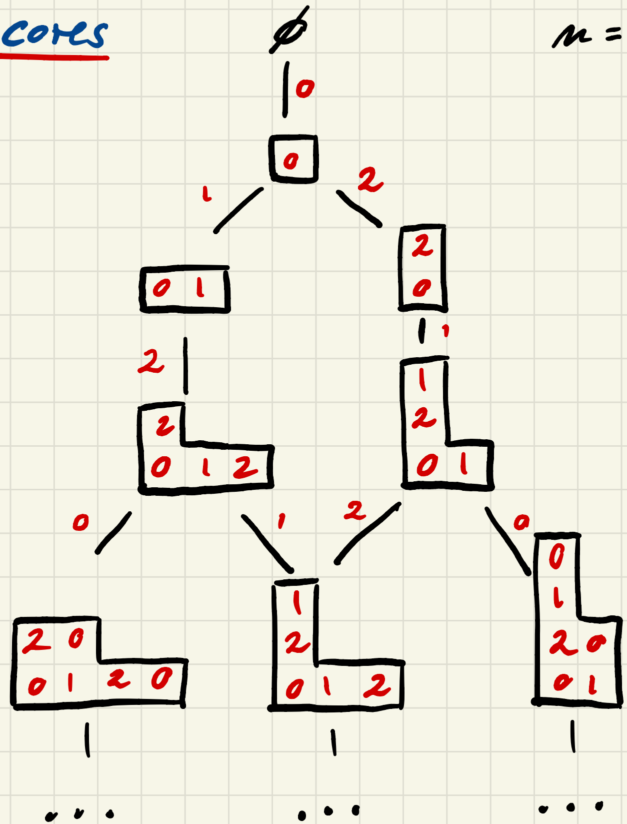


Lattice of n-cores

$n = 3$



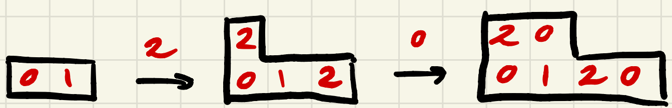
Definition A weak horizontal strip of size $r < n$ is a skew shape κ/τ of n-cores κ and τ s.t.

$$\tau \to \tau^{(1)} \to \tau^{(2)} \to \cdots \to \tau^{(r)} = \kappa$$

is a saturated chain in the n-core lattice.

Remark In particular, $\exists$ exactly r residues in κ/τ

Example $n=3$



weak horizontal 2-strip

$\longleftrightarrow$ $s_0 s_2$ cyclically decreasing

$$\boxed{\text{Weak horizontal } r\text{-strip} \longleftrightarrow \text{Cyclically decreasing factor of size } r}$$

Example

$w = s_0 s_2 s_1 s_0 \in \tilde{S}_3$

cyclically decreasing factorization:

$(02)(10)$

$(0)(2)(10)$ $(0)(21)(0)$ $(02)(1)(0)$

$(0)(2)(1)(0)$

$$\Rightarrow \tilde{F}_w = m_{22} + m_{211} + m_{1111}$$

Symmetric functions

homogeneous symmetric fct $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$

$$h_r = \sum_{1 \le i_1 \le i_2 \le \dots \le i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

monomial symmetric fct $m_\lambda = \sum_{\mathrm{sort}(\alpha) = \lambda} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{\ell(\alpha)}^{\alpha_{\ell(\alpha)}}$

$\Lambda_{(n)}$ and $\Lambda^{(n)}$

Λ ring of symmetric fct

$\Lambda_{(n)} \subseteq \Lambda$ subring generated by $h_1, h_2, \dots, h_{n-1}$

basis $\{h_\lambda \mid \lambda_1 < n\}$

$\Lambda^{(n)} = \Lambda / I_{(n)}$ $\quad I_n$ = ideal generated by m_μ with $\mu_1 \geq n$

basis $\{m_\lambda \mid \lambda_1 < n\}$

h_λ, m_μ are dual under the Hall inner product

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

Affine Schur functions

Definition $w \in \tilde{S}_n$ is an *affine Grassmannian permutation* if every reduced word of w ends in 0.

Remark Affine Grassmannian permutations in $\tilde{S}_n$ are in bijection with n-cores.
$\lambda(w)$ = n-core associated with w.

Definition $\tilde{F}_\lambda := \tilde{F}_w$ for $w \in \tilde{S}_n$ affine Grassmannian, $\lambda = \lambda(w)$

affine Schur fct

Theorem $\{\tilde{F}_\lambda \mid \lambda \text{ } n\text{-core}\}$ forms a basis of $\Lambda^{(n)}$.

Pf $\exists$ bijection b/w n-cores and $(n-1)$-bounded partitions
$\tilde{F}_\lambda$ triangular in monomial basis

k - Schur fcts $(k = n-1)$

$\tilde{F}_\lambda$ basis of $\Lambda^{(n)}$

k - Schur fct $s_\lambda^{(k)}$ is dual to $\tilde{F}_\mu$ under Hall inner product

$$\langle \cdot , \cdot \rangle : \Lambda_{(n)} \times \Lambda^{(n)} \to \mathbb{Q}$$

$$\langle s_\lambda^{(k)}, \tilde{F}_\mu \rangle = \delta_{\lambda\mu}$$

Theorem (Lam) Affine Stanley symmetric functions $\tilde{F}_w$ expand positively in terms of affine Schur fcts $\tilde{F}_\lambda$.

Theorem (Meroc, S.) $\forall w \in \tilde{S}_n \ \exists \lambda, \mu$ s.t. $\tilde{F}_w = \tilde{F}_{\lambda/\mu}$

NilCoxeter Algebra and Fomin-Stanley construction

Definition nilCoxeter algebra $\mathcal{A}_0$ algebra over $\mathbb{Z}$
generated by $\{A_i \mid 0 \le i < n\}$

$$A_i^2 = 0$$

$$A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1}$$

$$A_i A_j = A_j A_i$$

$A_w = A_{i_1} A_{i_2} \cdots A_{i_\ell}$ if $i_1 \ldots i_\ell$ is a reduced word for w

$$A_w \cdot A_v = \begin{cases} A_{wv} & \text{if } \ell(wv) = \ell(w) + \ell(v) \\ 0 & \text{else} \end{cases}$$

Definition $\mathbb{h}_k = \sum_{\substack{w \text{ cyclically} \\ \text{decreasing} \\ \ell(w)=k}} A_w$

Theorem $\mathbb{h}_1, \mathbb{h}_2, \ldots, \mathbb{h}_{n-1}$ commute

affine Stanley symmetric fct $\tilde{F}_w = \sum_{\substack{\alpha \\ \text{composition}}} \langle \mathbb{h}_{\alpha_\ell} \cdots \mathbb{h}_{\alpha_1}, A_w \rangle x^\alpha$

where $\langle A_w, A_v \rangle = \delta_{wv}$