Recall: Rough outline for proving Schur/key-positivity of Nonsymmetric Catalan Function

✓ ① Relate "character" of AGD-crystals to Nonsymmetric Catalan Functions

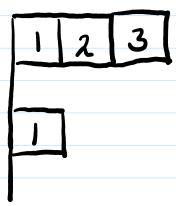
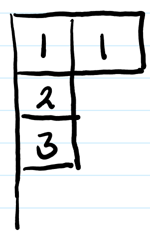
② Relate AGD crystals to DARK crystals

③ Decompose DARK crystals into a disjoint union of h.w. $U_q(\mathfrak{gl}_\ell)$ crystals / Demazure crystals

Notation: $\mathrm{Tab}_\ell(\mu)$ - set of all tabloids w/ length at most ℓ and content μ

$\cup\!\!\mid$

$\mathrm{SSYT}_\ell(\mu)$ - set of all semistandard Young tableaux w/ length at most ℓ and content μ

Ex:  $\in \mathrm{Tab}_4(2,1,1)$  $\in \mathrm{SSYT}_4(2,1,1)$

Def: Single-row Kirillov-Reshetikhin (KR) crystals $B^{1,s}$

$U_q'(\hat{\mathfrak{sl}}_\ell)$ - seminormal crystal

elements: all weakly increasing words of length s in alphabet $[\ell]$

crystal operators: $\tilde{e}_i(b)$ - change leftmost $i+1$ to i $\quad i \in [\ell-1]$

$\tilde{f}_i(b)$ - change rightmost i to $i+1$

$\tilde{e}_0(b)$ - remove a 1 from the beginning, add ℓ to end (if no 1, $\tilde{e}_0(b)=0$)

$\tilde{f}_0(b)$ - remove a ℓ from end, add 1 to the beginning (if no ℓ, $\tilde{f}_0(b)=0$)

[illegible]
beginning (if no ℓ, $\tilde{f}_0(b) = 0$)

Ex: $b = 1123 \in B^{1,4}$ $\quad\quad \ell = 3$

$\tilde{e}_2(b) = 1122$ $\quad\quad \tilde{f}_0(b) = 1112$

Def: $\mu = (\mu_1 \geq \dots \geq \mu_p \geq 0)$ $\quad B^\mu = B^{1,\mu_p} \otimes B^{1,\mu_{p-1}} \otimes \dots \otimes B^{1,\mu_1}$

elements in B^μ: biwords $b = \begin{pmatrix} v_1 & \dots & v_m \\ w_1 & \dots & w_m \end{pmatrix}$

- $v_i, w_i \in \mathbb{Z}_{\geq 1}$
- $i < j \quad v_i \geq v_j$
- $v_i = v_j \quad w_i \leq w_j$

crystal operators:

$$b = \rightarrow \begin{pmatrix} \overbrace{3333}^{4} & \overbrace{22222}^{5} & \overbrace{11111}^{5} \\ 2234 & 22333 & 11122 \end{pmatrix} \in B^{554} \quad \ell = 4$$

$i=2$ $\quad$)) ($\quad$)) ((($\quad$))

$\tilde{e}_i(b)$: leftmost unpaired $i+1 \to i$ $\quad \tilde{e}_2(b) = \begin{pmatrix} 3333 & 22222 & 11111 \\ 2234 & 22233 & 11122 \end{pmatrix}$

$\tilde{f}_i(b)$: rightmost unpaired $i \to i+1$ $\quad \tilde{f}_2(b) = \begin{pmatrix} 3333 & 22222 & 11111 \\ 2234 & 23333 & 11122 \end{pmatrix}$

Note: $b \in B^\mu \longleftrightarrow T \in \mathrm{Tab}_\ell(\mu)$

$\begin{pmatrix} 33 & 222 & 111 \\ 13 & 234 & 122 \end{pmatrix} \longleftrightarrow$

1	2	2
2	3	4
1	3	

Def: $b \in B^\mu \xleftrightarrow{\text{RSK}} (P(b), Q(b))$

$P(b)$: column insertion right to left using bottom(b)

$Q(b)$: recording tableau

Ex: $\begin{pmatrix} 33 & 222 & 111 \\ 13 & 234 & 122 \end{pmatrix} \rightarrow$

Insertion	Recording
122 4	111 2
122 34	111 22
122 234	111 222
⋮	⋮

$$\begin{array}{c} 1\,2\,2 \\ 2\,3\,4 \\ \vdots \\ 1\,1\,2\,2 \\ 2\,3\,4 \\ 3 \end{array} = P(b) \quad \Bigg| \quad \begin{array}{c} 2\,2\,2 \\ \vdots \\ 1\,1\,1\,3 \\ 2\,2\,2 \\ 3 \end{array} \; Q(b)$$

Thm: (Shimozono)?

$$\mathcal{B}^\mu = \bigsqcup_{T \in SSYT_\ell(\mu)} C_T \quad , \quad C_T = \{ b \in \mathcal{B}^\mu \mid Q(b) = T \} \cong \mathcal{B}^{\mathfrak{gl}}(sh(T))$$

h.w. $U_q(\mathfrak{gl}_\ell)$-crystal w/ h.w. $sh(T)$

Def: Partial insertion P_i

$T \in Tab_\ell$ $\qquad P_i(T)$ - tabloid formed by replacing rows i and $i+1$ w/ $P(T^{i+1}\, T^i)$

Ex: $$P_2\left(\begin{array}{l|l} & 1\,1\,2\,2 \\ \rightarrow & 1\,2\,3\,3\,4 \\ \rightarrow & 1\,1\,2\,2\,2\,3 \\ & 2\,3\,4 \end{array} \right) = \begin{array}{l} 1\,1\,2\,2 \\ 1\,1\,1\,2\,2\,2\,2\,3\,3\,4 \\ 2\,3 \\ 2\,3\,4 \end{array}$$

$$P(112223 \;\; 12334) = \begin{array}{l} 1\,1\,1\,2\,2\,2\,2\,3\,3\,4 \\ 2\,3 \end{array}$$

Def: inv : $\mathcal{B}^\mu \to \mathcal{B}^\mu$

inv(b) - biword formed by switching top(b) and bot(b) then sorting appropriately

Ex: $$b = \begin{pmatrix} 3\,3 & 2\,2\,2 & 1\,1\,1 \\ 1\,3 & 2\,3\,4 & 1\,2\,2 \end{pmatrix} \qquad \text{inv}(b) = \begin{pmatrix} 4 & 3\,3 & 2\,2\,2 & 1\,1 \\ 2 & 2\,3 & 1\,1\,2 & 1\,3 \end{pmatrix}$$

Prop: $b \in \mathcal{B}^\mu$ $\qquad T = \text{inv}(b)$

Then b is a h.w. element iff one of the following holds

a) $\tilde{e}_i(b) = 0 \qquad \forall\, i \in [\ell-1]$

b) $P_i(T) = T \quad \forall i \in [\ell-1]$

c) $T \in SSYT_\ell(\mu)$

<u>Def</u>: K-R affine Demazure (DARK) crystal

$\mu = (\mu_1 \geq \dots \geq \mu_p \geq 0) \qquad \underline{\omega} = (\omega_1, \dots, \omega_p) \in (\mathcal{H}_\ell)^p$

$$B^{\mu;\underline{\omega}} = \mathfrak{F}_{\omega_1}(\mathfrak{F}_\tau \mathfrak{F}_{\omega_2}(\dots(\mathfrak{F}_\tau \mathfrak{F}_{\omega_{p-1}}(\mathfrak{F}_\tau \mathfrak{F}_{\omega_p}\{b_{\mu_p}\} \otimes b_{\mu_{p-1}})\dots \otimes b_{\mu_2}) \otimes b_{\mu_1})$$

$b_s = 1^s \in B^{1,s}$ $\qquad$ $\mathfrak{F}_{\omega_i}$ - Demazure operator

$\mathfrak{F}_\tau$ - add 1 mod ℓ to each letter then sort

<u>Def</u>: kat : $\mathrm{Tab}_\ell \to \mathrm{Tab}_\ell$

kat(T) a) remove all 1's from T, left-adjust

b) remove 1^{st} row and add as ℓ^{th} row

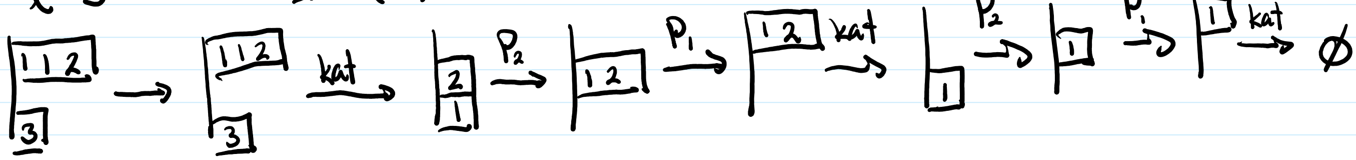
c) subtract 1 from all letters

<u>Def</u>: $\underline{\omega}$-katabolizable $\qquad \underline{\omega} = (\omega_1, \dots, \omega_p) \in (\mathcal{H}_\ell)^p$

$T \in \mathrm{Tab}_\ell$ $\underline{\omega}$-katabolizable if
- all 1's in $P_{\omega_1^{-1}}(T)$ in 1^{st} row
- $\mathrm{kat}(P_{\omega_1^{-1}}(T))$ is $(\omega_2, \dots, \omega_p)$-kat.

Only $\emptyset$-kat. tabloid is empty tabloid

<u>Ex</u>: $\ell = 3 \qquad \underline{\omega} = (id, s_2 s_1, s_2 s_1)$

T =


<u>Thm</u>: $\mu = (\mu_1 \geq \dots \geq \mu_p \geq 0) \qquad \underline{\omega} = (\omega_1, \dots, \omega_p) \in (\mathcal{H}_\ell)^p$

Thm: $\mu = (\mu_1 \geq \dots \geq \mu_p \geq 0)$ $\quad \underline{\omega} = (\omega_1, \dots, \omega_p) \in (\mathcal{H}_\ell)^r$

$$\{ T \in \mathrm{Tab}_\ell(\mu) \mid T \ \underline{\omega}\text{-kat.} \} \xleftrightarrow{\text{inv}} \mathcal{B}^{\mu;\underline{\omega}}$$

$$\text{s.t.} \quad \text{shape}(T) \longleftrightarrow \text{content}(\text{inv}(T))$$

Def: charge

unique function : words w/ partition content $\longrightarrow \mathbb{Z}_{\geq 0}$ s.t.

C1) $\text{charge}(\emptyset) = 0$

C2) $\text{content}(u) = \lambda$, $u = v1^{\lambda_1} \Rightarrow \text{charge}(u) = \text{charge}(v^-)$

(v^- subtract 1 from every number in v)

C3) $\text{content}(u) = \lambda$, $x \neq 1$ letter, $u = vx \Rightarrow \text{charge}(u) = \text{charge}(xv) + 1$

C4) charge constant on Knuth equiv classes.

Let $T \in \mathrm{Tab}_\ell(\mu)$. $\text{charge}(T) = \text{charge}(T^\ell T^{\ell-1} \dots T^1)$ $\quad T^i$ - i^{th} row of T

Ex:

$$\text{charge}\left(\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \right) = \text{charge}(43225113)$$

$$= \text{charge}(34322511) + 1$$

$$= \text{charge}(232114) + 1$$

$$= \text{charge}(423211) + 2$$

$$= \text{charge}(3121) + 2$$

$$= \text{charge}(3211) + 2$$

$$= \text{charge}(21) + 2$$

$$= \text{charge}(1) + 2 = \text{charge}(\emptyset) + 2 = 2$$

Prop: (Blasiak, 2020)

$\underline{\omega} = (\omega_1, \dots, \omega_p) \in (\mathcal{H}_\ell)^p$ $\qquad \mu = (\mu_1 \geq \dots \geq \mu_p \geq 0)$

$\underline{\omega} = (\omega_1, \ldots, \omega_p) \in (\mathcal{H}_\ell)^p \qquad \mu = (\mu_1 \geq \ldots \geq \mu_p \geq 0)$

$\exists$ isomorphism $\Theta : B^{\mu;\underline{\omega}} \otimes \mathcal{U}_{\mu,\Lambda_0} \longrightarrow AGD(\mu;\underline{\omega})$

Prop:
$$\pi_{\omega_1} x_1^{\mu_1} \Phi \pi_{\omega_2} x_1^{\mu_2} \cdots \Phi \pi_{\omega_p} x_1^{\mu_p} = q^{-n_\ell(\mu)} \operatorname{char}_{x;\mu}(AGD(\mu;\underline{\omega}))$$
$$= \sum_{b \in B^{\mu;\underline{\omega}}} q^{\operatorname{charge}(\operatorname{inv}(b))} x^{\operatorname{content}(b)}$$
$$= \sum_{\substack{T \in \operatorname{Tab}_\ell(\mu) \\ T\ \underline{\omega}\text{-kat.}}} q^{\operatorname{charge}(T)} x^{\operatorname{shape}(T)}$$

Prop: $\underline{\omega} = (\omega_1, \ldots, \omega_p)$ s.t. $\omega_1 = \omega_0$ longest element in $\mathcal{H}_\ell$
$$B^{\mu;\underline{\omega}} = \bigsqcup_{\substack{U \in SSYT_\ell(\mu) \\ U\text{-}(id, \omega_2, \ldots, \omega_p)\text{-kat}}} C_U$$

$C_U = \{ b \in B^\mu \mid Q(b) = U \}$ h.w. $\mathcal{U}_q(\mathfrak{gl}_\ell)$-crystal

Corollary: Ψ root ideal $\mu = (\mu_1 \geq \ldots \geq \mu_\ell \geq 0)$ $\underline{\omega} = (\omega_0, s(n_1), \ldots, s(n_{\ell-1}))$
$$H(\Psi, \mu, \omega_0)(\underline{x}; q) = \pi_{\omega_0} x_1^{\mu_1} \Phi \pi_{s(n_1)} x_1^{\mu_2} \cdots \Phi \pi_{s(n_{\ell-1})} x_1^{\mu_\ell}$$
$$= \sum_{b \in B^{\mu;\underline{\omega}}} q^{\operatorname{charge}(\operatorname{inv}(b))} x^{\operatorname{content}(b)}$$
$$= \sum_{\substack{U \in SSYT_\ell(\mu) \\ U\text{-}(id, s(n_1), \ldots, s(n_\ell))\text{-kat}}} q^{\operatorname{charge}(U)} s_{\operatorname{shape}(U)}$$

Prop:
$$B^{\mu;\underline{\omega}} = \bigsqcup_{\substack{T \in \operatorname{RowFrank}_\ell(\mu) \\ T \text{ extreme } \underline{\omega}\text{-kat}}} \tilde{C}_T$$

$\tilde{C}_T = \{ b \in B^{\mu;\underline{\omega}} \mid Q(b) = P(T) \} \cong \mathcal{F}_{\sigma(\operatorname{shape}(T))} \{ \mathcal{U}_{\operatorname{sort}(\operatorname{sh}(T))} \}$

$$\tilde{C}_T = \{ b \in \mathcal{B}^{\mu;\omega} \mid Q(b) = P(T) \} \cong \mathcal{J}_{p(\mathrm{shape}(T))} \{ u_{\mathrm{sort}(\mathrm{sh}(T))} \}$$

$\hookrightarrow U_q(\mathfrak{gl}_\ell)$ - Demazure crystal

<u>Corollary</u>: (Ψ, μ, ω) - tame root ideal s.t. $\mu = (\mu_1 \geq \dots \geq \mu_\ell \geq 0)$

$\underline{\omega} = (\omega, s(n_1), \dots, s(n_{\ell-1}))$

$$H(\Psi, \mu, \omega)(x, q) = \sum_{\substack{T \in \mathrm{RowFrank}_\ell(\mu) \\ T \text{ extreme } \underline{\omega}\text{-kat}}} q^{\mathrm{charge}(T)} \, K_{\mathrm{shape}(T)}$$

<u>Def</u>: $\mathrm{RowFrank}_\ell(\mu) = \{ T \in \mathrm{Tab}_\ell(\mu) \mid \mathrm{shape}(T)$ is a rearrangement of shape $P(T) \}$

<u>Def</u>: $T \in \mathrm{Tab}_\ell(\mu)$ $i \in [\ell - 1]$

- $S_i' = \mathrm{inv} \circ S_i \circ \mathrm{inv}(T)$
- $S_{ij}' = \mathrm{inv} \circ S_{ij} \circ \mathrm{inv}(T)$ $S_{ij} = S_i S_{i+1} \cdots S_{j-2} S_{j-1} S_{j-2} \cdots S_i$

<u>Def</u>: $T \in \mathrm{RowFrank}_\ell(\mu)$ is extreme $\underline{\omega}$-kat if

- T is $\underline{\omega}$-kat.
- $S_{ij}'(T)$ is not $\underline{\omega}$-kat $\forall\, i < j$ s.t. $\mathrm{sh}(T) < s_{ij}(\mathrm{sh}(T))$ is a covering relation in Bruhat order on $\mathbb{Z}^\ell$

Bruhat order on $\mathbb{Z}^\ell$: $\alpha < \beta$ if $\mathrm{sort}(\alpha) = \mathrm{sort}(\beta)$ and $p(\alpha) < p(\beta)$ in Bruhat order on S_ℓ

$p(\alpha)$ - shortest element sending α to $\mathrm{sort}(\alpha)$.