Work from "Demazure Crystals and the Schur Positivity of Catalan Functions" by Blasiak, Morse, Pun

- Aim: Schur and key-positivity of nonsymmetric Catalan Functions

1) Nonsymmetric Catalan Function:

Def: Symmetric group S_ℓ

gen: $s_1, \dots, s_{\ell-1}$

relations: (1) $s_i^2 = 1$

(2) $s_i s_j = s_j s_i$ $\quad |i-j|>1$

(3) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

Def: 0-Hecke Monoid of S_ℓ $\quad H_\ell$

gen: $\sigma_1, \dots, \sigma_{\ell-1}$

relations:
- $\sigma_i^2 = \sigma_i$
- (2)-(3) replace s_i w/ σ_i

- $\mathbb{Z}[q][\underline{x}] = \mathbb{Z}[q][x_1, \dots, x_\ell]$

Def: Demazure operator π_i

$f \in \mathbb{Z}[q][\underline{x}]$ $\qquad \pi_i(f) = \dfrac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}$

Extend this to $\pi_w = \pi_{i_1} \pi_{i_2} \dots \pi_{i_m}$ $\qquad w = \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_m} \in H_\ell$

reduced expression

Def: Key polynomials

$\alpha \in \mathbb{Z}^\ell$ $\qquad \kappa_\alpha = \pi_{\rho(\alpha)} x^{\mathrm{sort}(\alpha)}$ $\qquad \rho(\alpha) \in H_\ell$ of shortest

$\alpha \in \mathbb{Z}^\ell$ $\qquad$ $\kappa_\alpha = \pi_{p(\alpha)} x^{\mathrm{sort}(\alpha)}$ $\qquad$ $p(\alpha) \in H_\ell$ of shortest length $\alpha \to \mathrm{sort}(\alpha)$

Prop: (Reiner-Shimozono, 1995)

$\{\kappa_\alpha \mid \alpha \in \mathbb{Z}^\ell\}$ form a basis for $\mathbb{Z}_q[\underline{x}]$ and $\mathbb{Z}_q[\underline{x}^{\pm 1}]$

Def: Polynomial truncation operator poly

$$\mathrm{poly}(\kappa_\alpha) = \begin{cases} \kappa_\alpha & \text{if } \alpha \in \mathbb{Z}^\ell_{\geq 0} \\ 0 & \text{else} \end{cases}$$

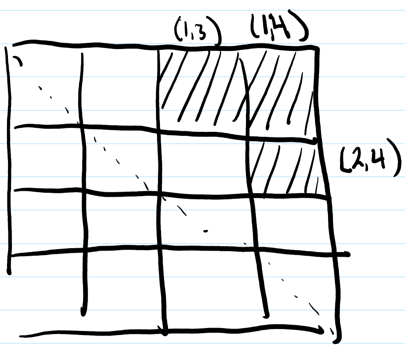
Def: Root ideal ψ

upper order ideal of poset $(\Delta^+_\ell, \leq)$

$$\Delta^+_\ell = \{(i,j) \mid 1 \leq i < j \leq \ell\}$$

$$(a,b) \leq (c,d) \iff a \geq c \text{ and } b \leq d$$

Ex: $\ell = 4$



$\psi = \{(1,3), (1,4), (2,4)\}$

Def: Nonsymmetric Catalan Function $H(\psi, \gamma, w)(\underline{x}; q)$

$\psi \in \Delta^+_\ell$, $w \in H^\ell$, $\gamma \in \mathbb{Z}^\ell$

$$H(\psi, \gamma, w) = \pi_w\Big(\mathrm{poly}\Big(\Big(\prod_{(i,j)\in\psi} \Big(1 - q\frac{x_i}{x_j}\Big)^{-1}\Big) \cdot x^\gamma\Big)\Big)$$

Ex: $\ell = 4$ $\quad$ ψ as above

$$H(\psi, (2,3,0,0), \sigma_1) = \pi_1 \, \mathrm{poly}\Big(\Big(1 + q\frac{x_1}{x_3} + q^2\frac{x_1^2}{x_3^2} + \dots\Big)\Big(1 + q\frac{x_1}{x_4} + \dots\Big)\Big(1 + q\frac{x_2}{x_4} + \dots\Big) x_1^2 x_2^3\Big)\Big)$$

$$H(\psi, (2,3,0,0), \sigma_1) = \pi_1\left(\text{poly}\left(\left(1 + q\frac{x_1}{x_3} + q^2\frac{x_1^2}{x_3^2} + \dots\right)\left(1 + q\frac{x_1}{x_4} + \dots\right)\left(1 + q\frac{x_2}{x_4} + \dots\right) x_1^2 x_2^3\right)\right)$$
$$= \pi_1\left(\text{poly}\left(x_1^2x_2^3 + q\left(x_1^3x_2^3x_3^{-1} + x_1^3x_2^3x_4^{-1} + x_1^2x_2^3x_4^{-1}\right) + q^2(\dots) + \dots\right)\right)$$
$$= \pi_1\left(\text{poly}\left(K_{2,3} - K_{3,2} + q\left(K_{3,3,-1,0} - K_{3,3,0,-1} + K_{3,3,0,-1} + K_{2,4,0,-1} - K_{4,2,0,-1} - K_{3,3,0,-1}\right) + \dots\right)\right)$$
$$= \pi_1(K_{2,3} - K_{3,2}) = K_{3,2} - K_{2,3}$$

Remark: if $\omega = \omega_0$ (longest element in H_ℓ) then $H(\psi, \gamma, \omega)$ reduces to Symmetric Catalan function

Def: $n(\psi)$

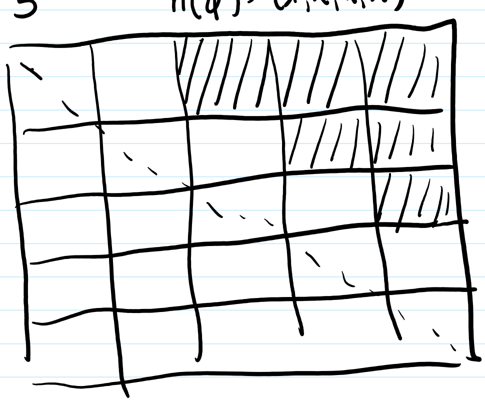
$\psi \in \Delta_\ell^+$ $\qquad n(\psi) = (n_1, \dots, n_{\ell-1}) \in [\ell]^{\ell-1}$

$$n_i = \left|\{ j \in \{i, \dots, \ell\} : (i,j) \notin \psi \}\right|$$

Def: Tame labeled root ideal

(ψ, γ, ω) is tame if $\quad \{n_1+1, \dots, \ell-1\} \subseteq \{ j \in [\ell-1] \mid \omega\sigma_j = \omega \}$

Ex: $\ell = 5$ $\qquad n(\psi) = (2,2,2,2)$ $\qquad \omega = \sigma_3\sigma_4\sigma_3$ $\qquad \gamma = (2,2,2,1,1)$



$\{2+1, 5-1\} = \{3,4\}$

$\omega\sigma_3 = \sigma_3\sigma_4\sigma_3\sigma_3 = \sigma_3\sigma_4\sigma_3 = \omega$

$\omega\sigma_4 = \sigma_3\sigma_4\sigma_3\sigma_4 = \sigma_3\underset{\parallel}{\sigma_3}\sigma_4\sigma_3$

Def: $\quad \Phi : \mathbb{Z}[q][\underline{x}] \to \mathbb{Z}[q][\underline{x}]$

$\Phi(x_i) = x_{i+1} \quad i \in [\ell-1] \qquad \Phi(x_\ell) = q x_1$

Prop: (ψ, γ, ω) tame labeled root ideal, $\gamma \in \mathbb{Z}_{\geq 0}^\ell$

$$H(\psi', \gamma, w) = \pi_w X_1^{\gamma_1} \Phi \, \pi_{s(n_1)} X_1^{\gamma_2} \Phi \cdots \Phi \, \pi_{s(n_{\ell-1})} X_1^{\gamma_\ell}$$

where $s(d) = \sigma_{\ell-1} \cdots \sigma_d \in H_\ell$

2) <u>Crystals</u>:

$\mathfrak{g}$ - sym. Kac-Moody Lie Algebra $\qquad U_q(\mathfrak{g})$

- I - nodes for Dynkin diagram
- P^* - coweight lattice $\qquad \{\alpha_i^\vee\} \subseteq P^*$ coroots
- P - weight lattice $\qquad \{\alpha_i\} \subseteq P$ roots

$\langle \cdot , \cdot \rangle : P \times P \to \mathbb{Q}$

<u>Def</u>: $U_q(\mathfrak{g})$ seminormal crystal is a set B w/

- $\mathrm{wt} : B \to P$
- $f_i, e_i : B \sqcup \{0\} \to B \sqcup \{0\}$ $\qquad f_i(0) = e_i(0) = 0 \qquad i \in I$

s.t. 1) $\mathrm{wt}(e_i b) = \mathrm{wt}(b) + \alpha_i$

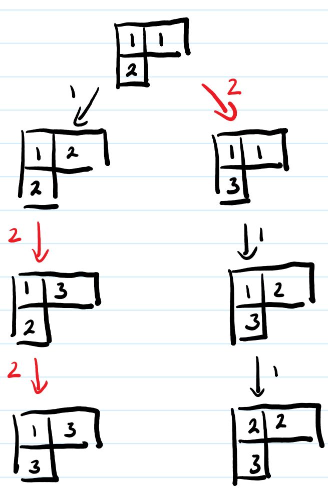
$\mathrm{wt}(f_i b) = \mathrm{wt}(b) - \alpha_i$

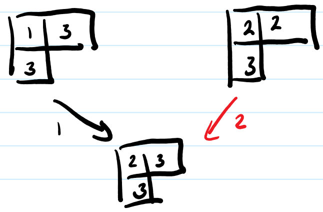
2) $\varepsilon_i(b) = \max\{k \geq 0 \mid e_i^k(b) \neq 0\} < \infty$

$\varphi_i(b) = \max\{k \geq 0 \mid f_i^k(b) \neq 0\} < \infty$

$\langle \alpha_i^\vee, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$

<u>Ex</u>: $U_q(\mathfrak{gl}_3)$ $\qquad$ crystal on SSYT





Def: character $\quad ch(B) = \sum_{b \in B} x^{wt(b)}$

Prop: Let $B(u_\lambda)$ be a h.w. $U_q(\mathfrak{gl}_\ell)$ crystal w/ h.w. λ

$$ch(B(u_\lambda)) = s_\lambda(\underline{x})$$

Def: Demazure operator $\mathfrak{J}_i$

B - $U_q(\mathfrak{g})$ seminormal crystal $\qquad S \subseteq B$

$$\mathfrak{J}_i S = \{ f_i^m b \mid b \in S;\ m \geq 0 \} \setminus \{0\} \subseteq B$$

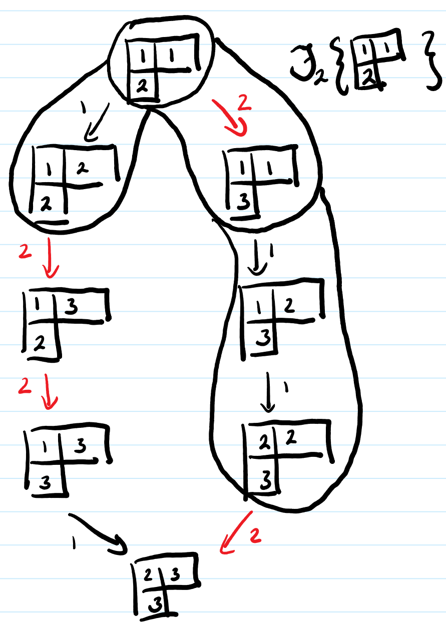
Def: $U_q(\mathfrak{g})$-Demazure crystal subset of h.w. $U_q(\mathfrak{g})$ crystal $B(\Lambda)$ of the form

$$\mathfrak{J}_{i_1} \cdots \mathfrak{J}_{i_k} \{ u_\Lambda \}$$

$\uparrow$ h.w. element of $B(\Lambda)$

Ex:

$\mathfrak{J}_1 \mathfrak{J}_2 \left\{ \begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix} \right\}$



Prop: $\alpha \in \mathbb{Z}^{\ell}_{\geq 0}$ $\quad B(u_{\dots}, \alpha)$ - h.w. $U_q(\mathfrak{gl}_\ell)$ crystal w/ h.w. sort(α)

Prop: $\alpha \in \mathbb{Z}^{\ell}_{\geq 0}$ $\quad B(u_{\mathrm{sort}(\alpha)})$ - h.w. $U_q(\mathfrak{gl}_\ell)$ crystal w/ h.w. $\mathrm{sort}(\alpha)$

$$\mathrm{ch}(\mathfrak{F}_{p(\alpha)}(u_{\mathrm{sort}(\alpha)})) = \kappa_\alpha(\underline{x})$$

Rough Outline: $\cdot$ "char"$(\underbrace{\text{AGD-crystal}}_{\hookrightarrow \text{ not nice}}) = \pi_{\omega_1} x_1^{\mu_1} \Phi \pi_{\omega_2} x_1^{\mu_2} \Phi \ldots \Phi \pi_{\omega_p} x_1^{\mu_p}$

- AGD crystal $\xleftrightarrow{\text{"isomorphism"}}$ DARK crystal
- DARK crystal $\leftarrow$ decompose this into a disjoint union of $U_q(\mathfrak{gl}_\ell)$ - Demazure crystals

Def: Extended Affine Symmetric Group $\tilde{S}_\ell$

gen: $s_i \quad i \in \mathbb{Z}/\ell\mathbb{Z}$, τ

relations: (i) $s_i^2 = 1$

(ii) $s_i s_j = s_j s_i \quad i \notin \{j-1, j+1\}$

(iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

(iv) $\tau s_i = s_{i+1} \tau$

(v) $\tau^\ell = 1$

Def: 0-Hecke Monoid $\tilde{H}_\ell$

gen: $\sigma_i \quad i \in \mathbb{Z}/\ell\mathbb{Z} \quad \tau$

rel: $\cdot$ $\sigma_i^2 = \sigma$

$\cdot$ (ii) – (v) replace s_i w/ σ_i

Def: $B(\Lambda)$ - h.w. $U_q(\hat{\mathfrak{sl}}_\ell)$ crystal h.w. Λ

h.w. element u_Λ

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$$\mathfrak{T}_\tau : B(\Lambda) \to B(\tau(\Lambda))$$

$$f_{j_1}^{d_1} \cdots f_{j_k}^{d_k} (u_\Lambda) \longrightarrow f_{\tau(j_1)}^{d_1} \cdots f_{\tau(j_k)}^{d_k} (u_{\tau(\Lambda)})$$

Def: $U_q(\hat{sl}_\ell)$ - generalized Demazure (GD) crystal

$\Lambda_1, \ldots, \Lambda_p$ dominant weights $\quad \underline{\omega} = (\omega_1, \ldots, \omega_p) \in (\tilde{\mathcal{H}}_\ell)^p$

$$\mathfrak{T}_{\omega_1}(\mathfrak{T}_{\omega_2}(\cdots \mathfrak{T}_{\omega_{p-1}}(\mathfrak{T}_{\omega_p}\{u_{\Lambda_p}\} \otimes u_{\Lambda_{p-1}}) \otimes u_{\Lambda_{p-2}}) \cdots \otimes u_{\Lambda_1})$$

Def: Affine generalized Demazure (AGD) crystal

$$AGD(\mu; \underline{\omega}) = \mathfrak{T}_{\omega_1}(\mathfrak{T}_{\tau\omega_2}(\cdots \mathfrak{T}_{\tau\omega_{p-1}}(\mathfrak{T}_{\tau\omega_p}\{u_{\mu^p\Lambda_1}\} \otimes u_{\mu^{p-1}\Lambda_1}) \cdots \otimes u_{\mu^1\Lambda_1})$$

$\mu = (\mu_1 \geq \cdots \mu_p \geq 0) \qquad \underline{\omega} = (\omega_1, \ldots, \omega_p) \in (\mathcal{H}_\ell)^p$

$$\mu^i = \mu_i - \mu_{i+1}$$

Prop: $\pi_{\omega_1} x_1^{\mu_1} \Phi\, \pi_{\omega_2} x_1^{\mu_2} \cdots \Phi\, \pi_{\omega_p} x_1^{\mu_p} = q^{-n_\ell(\mu)} \operatorname{char}_{x;\mu}(AGD(\mu; \underline{\omega}))$

$$n_\ell(\mu) = \frac{|\mu|(\ell-1)}{2\ell} - \frac{1}{\ell}\sum_{i=1}^{p}(i-1)\mu_i$$