Meeting 12/4

Monday, January 4, 2021 10:33 AM

$Gr(k,n)$ = Grassmann variety = { k-planes in n-dimensional space }

$$k\left\{\underbrace{\begin{pmatrix} & V & \end{pmatrix}}_{n}\right.$$

$k \times n$ matrix of rank k

rows of V = basis in a k-plane.

$GL(k)$ acts by multiplication on the left = base change

$$\begin{pmatrix} | & | & | & | & & | \\ v_1 & v_2 & \cdots & \cdots & \cdots & v_n \end{pmatrix}$$

$v_1 \ldots v_n$ = columns of V

$\rightsquigarrow$ repeat periodically

$$v_1, v_2, v_3, \ldots v_n, v_1, v_2, \ldots v_n, \ldots$$

$$v_{i+n} = v_i$$

We're interested in ranks of various subsets in this sequence

$$f(i) = \min\{j : v_i \in \operatorname{Span}(v_{i+1}, \ldots v_j)\}$$

$$\underline{i+n \geq f(i) \geq i} \qquad f : \mathbb{Z} \to \mathbb{Z}$$

$v_{i+n} = v_i$ = if $v_i = 0$

$f(i+n) = f(i) + n$? $f(1) + \cdots + f(n)$

$v_{i+n} = v_i$

$f(i+n) = f(i) + n$

$f(1)+\dots+f(n) = kn$

Fact: f is a bijection! $\longrightarrow$ affine permutation

$\mathring{\Pi}_f = \{$all matrices V with fixed function $f\}$

positroid strata in $Gr(k,n)$

$$Gr(k,n) = \bigsqcup_f \mathring{\Pi}_f$$

When $\mathring{\Pi}_f$ is nonempty?

$\mathring{\Pi}_f$ are also enumerated by other combinatorial data:

- Rank matrix

  $r_{ij} = \text{rank}\{v_i \dots v_j\}$

- Pairs of permutations (u, w), $u \in S_n$, $w \in S_n$, satisfying some conditions: $u \le w$ in Bruhat order

  $w(1) \le \dots \le w(k)$

  $w(k+1) \le \dots \le w(n)$

  $f = u t_k w$, translation by k-th fundamental

- Juggling patterns...

<u>Thm</u> [KLS] $\mathring{\Pi}_f$ = projection of the Richardson variety $\mathring{X}_u^w$ under the map $Fl_n \longrightarrow G(k,n)$

the map $\mathrm{Flag}_n \xrightarrow{\ g\ } G(k,n)$

Thm (KLS) The cohomology class of the (closure of) $\mathring{\Pi}_f$ = affine Stanley function for f

Ex Maximal positroid cell $\mathring{\Pi}_{k,n} \subseteq G(k,n)$

$\mathrm{rk}(v_i, \dots, v_{i+k-1}) = k$ for all i

$$\underbrace{v_1 \cdots}_{k} \underbrace{\cdots v_n \, v_1 \, v_2}_{k \to \text{rank} = k.} \, v_3 \cdots$$

Thm [GL] $H^*(\mathring{\Pi}_{k,n})$ = rational q,t-Catalan number $C_{k,n}(q,t)$

q,t = homological grading + Hodge filtration $\quad GCD(k,n) = 1$

Also related to link invariants

$f \rightsquigarrow (u,w) \rightsquigarrow \beta(u) \cdot \beta(w)^{-1} \in$ braid group.

$H^*(\mathring{\Pi}_f)$ related to knot homology of $\beta(u) \cdot \beta(w)^{-1}$.

1/11	Erik	[KLS] Section 2-3 combinatorics	[Daniel]
1/18	MLK day	.	
1/25	Erik	[KLS] Sections 4-5	[V...]

1/25	Erik	[KLS] Sections 4-5 geometry + defn of positroid + properties [~~Yuze~~ Ashleigh]
2/1	Joseph (recap from last quarter)	
2/8	Daniel	
2/15	Presidents day	[KLS] Connection to affine Stanley [Wencin]
2/22		[GL] q,t - Catalan + knots (Eugene)
3/1		
3/8		