Informal ADM 3/1

Monday, March 1, 2021 11:00 AM

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Today: combinatorics (sections 2-3)

Next week: geometry (sections 4-5)

Def An affine permutation is a bijection $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(i+n) = f(i) + n$

all $f(i)$ in the window have different remainders mod n

Window notation $[f(1) f(2) \dots f(n)]$

Affine permutations form a group $\tilde{S}_n$ (extended affine S_n)

Def $f \in \hat{S}_n$ is k-bounded if

(a) $\frac{1}{n} \sum_{i=1}^{n} (f(i) - i) = k$ (fix average in the window)

(b) $\boxed{i \leq f(i) \leq i+n}$ for all i (sufficient to check in the window)

$0 \leq k \leq n$

Bound$(k,n) = \{k\text{-bounded affine permutations}\}$

Ex $n=4$ $f = [5\,2\,4\,7]$ $f(1)=5$ $f(2)=2$ $f(3)=4$ $f(4)=7$

$$k = \frac{1}{4}\left((5-1) + (2-2) + (4-3) + (7-4)\right)$$

$$= \frac{1}{4}(4+0+1+3) = 2$$

$1 \leq f(1) = 5 = 1+4$ $\quad 2 = f(2) < 2+4$ $\quad 3 < f(3) < 3+4$ $\quad 4 < f(4) < 4+4$

Rem Since $f(i) < i+n \leq 2n$ in the window.

Rem Since $f(i) < i+n \leq 2n$ in the window.
In fact, condition (a) means that there are exactly k values of i such that $\underbrace{n < f(i) \leq 2n}$
and exactly $n-k$ values of i such that $\underbrace{1 \leq f(i) \leq n}$.

Motivation

$$V = \begin{array}{c} \\ k \end{array}\underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}}_{n}$$

rank k
k matrix

Repeat v_i periodically such that $v_{i+n} = v_i$
Given a matrix V, we can define
a function $f(i) = \min_{j > i} \{ v_i \in \mathrm{Span}(v_{i+1}, \dots v_j) \}$

Since $v_{i+n} = v_i$, we have $f(i) \leq i+n$.

Clearly, $f(i) \geq i$ ($f(i) = i$ if $v_i = 0$)
Also, $f(i+n) = f(i) + n$ clearly.
Next time: we'll check that f is indeed a bijection
and it is k-bounded.

$\mathring{\Pi}_f$ = open positroid cell corresponding to f $= \left\{ V \text{ for fixed function } f \right\} / \text{row operations}$
$\cap$
$Gr(k,n)$.

Main object of KLS

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② Decorated permutations (Postnikov)

$f \in \mathrm{Bound}(k,n) \longrightarrow \bar{f} \in S_n$ such that

$f \in \check{B}ound(k,n) \longrightarrow \bar{f} \in S_n$ such that $\bar{f} = f \bmod n$.

Ex $f=[5\ 2\ 4\ 7] \longrightarrow \bar{f} = [\underset{+1}{1}\ \underset{-1}{2}\ 4\ 3]$

Is there a loss of information?

If $f(i) \neq i, i+n$, there is exactly one value in $(i, i+n)$ for each nonzero remainder.

$\Rightarrow$ can reconstruct $f(i)$ from $\bar{f}(i)$.

If $\bar{f}(i) = i$ then $f(i) = i$ or $i+n$.

Decoration: which fixed points of $\bar{f}$ (-1) correspond to i, and (1) if $i+n$.

Ex $\bar{f} = [\underset{+1}{1}\ \underset{-1}{2}\ 4\ 3] \longrightarrow f(1) = 1+4 = 5$ (decoration)

$f(2) = 2$ (decoration)

$3 < f(3) < 3+4$ and $f(3) \equiv 4 \bmod 4$

$\Rightarrow f(3) = 4$

$4 < f(4) < 8$ and $f(4) \equiv 3 \bmod 4$

$\Rightarrow f(4) = 7$.

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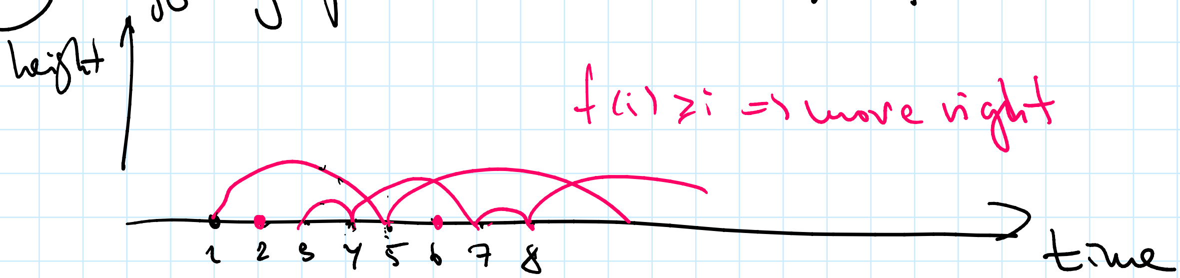
③ Juggling pattern $[5\ 2\ 4\ 7] = f$



$f(i) \geq i \Rightarrow$ move right

We see 2 connected trajectories (+ fixed points)

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Average condition ⇒ k trajectories.
Juggle with k balls, this is a graph of juggling.

④ Cyclic rank matrices

$$r_{ij} = \#\{x < i : f(x) > j\}$$

in the paper they used different notation $k - r_{ij}$

How to think about this? Draw a "graph" of f

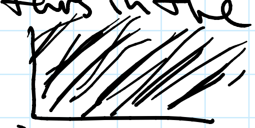
$j = f(i)$

$f = [5247]$

$i \backslash j$	1	2	3	4	5	6	7	8
1					*			
2		*						
3				*				
4							*	

$1 \le i \le n$

r_{ij} counts stars in the quadrant



(i,j)

$i \backslash j$	1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0
2	2	2	1	1	0	0	0	0
3	3	2	1	1	0	0	0	0
4	4	3	2	1	0	0	0	0

Properties of r_{ij}:

(1) $r_{i+n,j+n} = r_{ij}$

(2) $r_{ij} = 0$ if $j \geq i+n$

Indeed, if $r_{ij} \neq 0$, then

$$j < f(x) \leq x + n < i + n$$

(3) $r_{ij} = i - j + 1$ if $j < i$

$r_{ij} \leq k$

(All stars are above diagonal, for (i,j) below diagonal

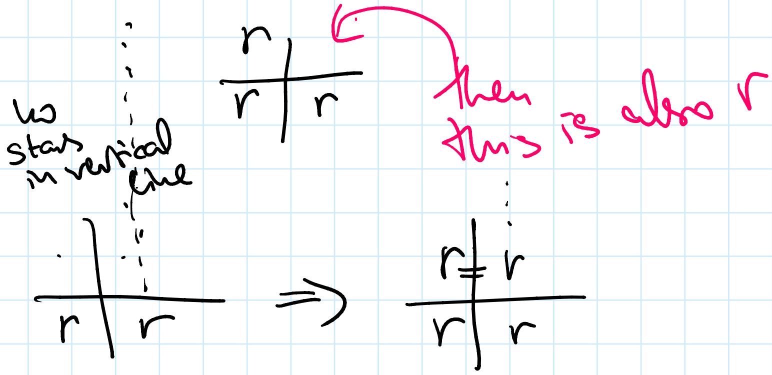
(4) $r_{i,j+1} = r_{ij}$ or $r_{ij} - 1$ (decrease in rows, at most by 1)

$r_{i+1,j} = r_{ij}$ or $r_{ij} + 1$ (increase in columns, at most by 1)

(5) "Matroid" if $r_{ij} = r_{i...}$ and

at most by 1)

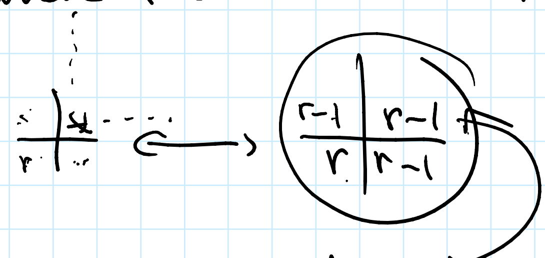
(5) "Matroid" if $r_{ij} = r_{i-1,j}$ and $r_{ij} = r_{i,j+1}$ then $r_{ij} = r_{i-1,j+1}$.



Claim: This is a bijection $\mathrm{Bound}(k,n) \longrightarrow (r_{ij})$ satisfying all above condition.

Given r_{ij}, how to reconstruct f?

We need to know where the stars are



$i \backslash j$	1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0
2	2	2	1	1	0	0	0	0
3	3	2	1	1	0	0	0	0
4	4	3	2	1	0	0	0	0
					1	1	0	

← by periodicity.

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⑤ Pairs of permutations

$(u, w) \in S_n$ • w is k-Grassmannian

$w(1) < \dots < w(k)$ def

$w(k+1) < \dots \dots < w(n)$

• $u \leq w$ in Bruhat order

Need affine permutation $t_k = [1+n, 2+n, \ldots k+n, k+1, \ldots n]$

Construction: $f = \boxed{f_{u,w} = u\, t_k\, w^{-1}}$ interpret w and u as affine perm.

Claim: This is a bijection, and $f_{u,w}$ is in Bound(k,n).

Given f, how to reconstruct u and w?

There are exactly k values $i_1 < \ldots < i_k$

such that $f(i_1) > n$, $f(i_2) > n \ldots f(i_k) > n$

exactly $n-k$ values $j_1 < \ldots < j_k$ such that

$f(j_s) \leq n$.

$w = [i_1\, i_2 \ldots i_k\, j_1 \ldots j_{n-k}]$.

$u(1) = f(i_1) - n \qquad u(k) = f(i_k) - n$

$u(2) = f(i_2) - n \quad \ldots \quad u(k+1) = f(j_1)$

$u(n) = f(j_{n-k})$.

Let's check: $u t_k w^{-1}(i_1) = u t_k(1) =$

$= u(1+n) =$

same for all i's

$= u(1) + n = f(i_1) - n + n$

$= f(i_1)$.

$u t_k w^{-1}(j_1) = u t_k(k+1)$

$= u(k+1) = f(j_1)$.

$= u(k+1) = f(j_1)$.

<u>Ex</u> $f = [5\,2\,4\,7]$ $\qquad i_1 = 1 \quad i_2 = 4$

$\qquad j_1 = 2 \quad j_2 = 3$

$w = [1\,4\,2\,3] \qquad w^{-1} = [1\,3\,4\,2]$

$s_3 s_2$

$u = [\underset{5-4}{\overset{\shortparallel}{1}}\ \underset{7-4}{\overset{\shortparallel}{3}}\ 2\,4] \quad s_2 \qquad t_k = [5\,6\,3\,4]$

This is one line notation?

$\underbrace{u}_{[1\,3\,2\,4\,5\,7\,6\,8]} t_k w^{-1} = \underset{\underset{6\to 7}{u}}{[1\,3\,2\,4]}\ \underset{t_k}{[5\,6\,3\,4]}\ \underset{w^{-1}}{[1\,3\,4\,2]} = [5\,2\,4\,7]$

Bergeron - Sottile $\Rightarrow$ f is bounded iff $u \leq w$ in Bruhat!