Informal ADM 3/29

Monday, March 29, 2021 10:52 AM

Open positroid variety $\mathring{\Pi}_f \subset Gr(k,n)$

defined by equivalent data:

① f = k-bounded affine permutation

② (r_{ij}) = cyclic rank matrix

③ $u \le w$, $u, w \in S_n$; w - k-Grass.

in Bruhat order

Last quarter: how to get between these descriptions.

$$k \left(\begin{matrix} | & & & | \\ v_1 & \cdots & \cdots & v_n \\ | & & & | \end{matrix} \right)$$
$$\underbrace{\hphantom{v_1 \cdots \cdots v_n}}_{n}$$

rank k, $k \times n$ matrix

$v_1 \dots v_n$ = columns

extend periodically $v_{i+n} = v_i$

$$\mathring{\Pi}_f = \{ V : \mathrm{rank}(v_i, \dots v_j) = r_{ij} \} \,/\, \text{row operations}$$

fixed cyclic rank matrix corresponding to f.

Π_f = closure of $\mathring{\Pi}_f$ in $Gr(k,n)$

$= \{ V : \mathrm{rank}(v_i, \dots v_j) \le r_{ij} \} /$

$= \{V : \mathrm{rank}(v_i \dots v_j) \le r_{ij}\} / \text{row gen.}$

Π_f = <u>closed</u> positroid variety

<u>Q</u>: Compute the class of $[\Pi_f]$ in (equivariant) cohomology of $Gr(k,n)$.

<u>A</u>: (next time?) In non-equivariant cohomology, $[\Pi_f]$ = affine Stanley symmetric fn.

<u>Today</u> An algorithm for computing $[\Pi_f]$ in equivariant cohomology $\Rightarrow$ by forgetting equivariant parameters, $[\Pi_f]$ in non-equivar.

Main tool = Richardson varieties

$\mathring{X}_w^u$ = open Richardson variety

$$\underset{Flag(n)}{\cap} \quad = \underbrace{X^w}_{\text{Schubert cell}} \cap \underbrace{X_u}_{\text{opposite Schubert cell}}.$$

Smooth non-empty if $u \le w$ ...

smooth, non-empty if $u \leq w$ in Bruhat order, $\dim = \ell(w) - \ell(u)$

X_w^u = closed Richardson variety = closure of $\mathring{X}_w^u$

$\pi : Flag(n) \longrightarrow Gr(k,n)$

$\{F_1 \subset F_2 \ldots \subset F_n\} \longrightarrow F_k$

<u>Last quarter:</u> $u \leq w$, w is k-Grassmannian

<u>Thm</u> $\pi(\mathring{X}_w^u) \simeq \mathring{\Pi}_f$ where f corresponds to the pair (u, w)

$\pi(X_w^u) \simeq \Pi_f$, similarly.

π is an isomorphism on X_w^u.

$T = (\mathbb{C}^*)^n$ acts on $Flag(n)$, $Gr(k,n)$, $\mathring{\Pi}_f$

$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ $t = (t_1 \ldots t_n) \in T$

$(v_1, \ldots v_n) \to (t_1 v_1, \ldots t_n v_n)$ preserves all rank conditions

Consider T-equivariant cohomology

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$$H_T^*(Gr(k,n)) \ni [\Pi_f].$$

$\boxed{\underline{\underline{\text{Prop 7.2}}}\ X_\square \cdot [\Pi_f] = (X_\square|_{\sigma(u)}) \cdot \Pi_f + \sum_{u \overset{\cdot}{<} u' \leq w} [\Pi_{f_{u',w}}]}$

poly in equiv parameters.

fixed point

where $X_\square$ = class of Schubert variety corresponding to $\square$ ($=_{\text{as symmetric fn}} p_1$)

$\sigma(u) = u(\{1,\dots k\})$

$\overset{\cdot}{<}$ = cover relation in Bruhat order

$u' = u \cdot t \leftarrow$ transposition

$\ell(u') = \ell(u) + 1.$

$\underline{\underline{\text{Proof}}}$: $X_\square \cdot [\Pi_f] = X_\square \cdot \pi_*([X_w^u]) \underset{\text{projection formula}}{=}$

$= \pi_*(\underbrace{\pi^*(X_\square) \cdot [X^w] \cdot [X_u]}_{\text{class in } H_T^*(Fl_n)})$

$= \pi_*([X_{s_k}] \cdot [X^w] \cdot [X_u])$ $\quad (*)$

Class of Schubert variety for simple reflection $s_k = (k\ k+1)$

Class A

Schubert variety for simple reflection $s_k = (k\ k+1)$

Monk's formula:

$$[X_{s_k}]\cdot[X_u] = [X_{s_k}]|_u \cdot [X_u] + \sum_{u \lessdot u'} [X_{u'}]$$

poly in equivariant parameters

Using Monk's formula, can rewrite (*):

$$\pi_*\left([X_{s_k}]|_u \cdot [X_u]\cdot[X^w]\right) + \sum_{u' \gtrdot u} \pi_*\left([X_{u'}]\cdot[X^w]\right)$$

$$= \pi_*\left([X_{s_k}]|_u \cdot \underbrace{[X^u_w]}_{\text{class of Richardson}}\right) + \sum_{u' \gtrdot u} \pi_*\left(\underbrace{[X^{u'}_w]}_{\text{vanishes unless } u' \leq w}\right)$$

$$= (X_D|_{\sigma(u)})\cdot[\Pi_f] + \sum_{u \lessdot u' \leq w} [\Pi_{u',w}].$$

Lemma 7.4 $[\Pi_f] \in H^*_T(Gr(k;n))$ is uniquely determined by the following

(1) $[\Pi_f]$ is homogeneous of degree

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$= \operatorname{codim} \Pi_f = k(n-k) - \ell(w) + \ell(u)$

(Recall $\dim \Pi_f = \ell(w) - \ell(u)$).

(2) Prop. 7.2 above ($\approx$ "Pieri rule")

(3) Point classes: if $w = u$

is a point, and it is $\sigma(u) = \sigma(w)$
$= w(\{1 \ldots k\})$.

fixed point of T-action in $Gr(k,n)$ $\Leftrightarrow$ k-element subset in $\{1 \ldots n\}$

$$[\Pi_{w,w}] = [\sigma(w)].$$

<u>Proof</u>: Induction by $\ell(w) - \ell(u)$

If $\ell(w) = \ell(u) \Rightarrow$ know

Lemma 7.2:

$$\underbrace{(x_D - x_D|_{\sigma(u)})}[\Pi_{u,w}] = \sum_{u \lessdot u' \leq w} [\Pi_{u',w}]$$

know by induction assumption

this restricts as $\neq 0$ poly in equiv. parameters

poly in equiv. parameters
to all fixed points but $\sigma(u)$

$\Rightarrow$ the difference is supported
at $\sigma(u)$, by degree reasons it is 0. ☒

April 5: Anne on affine Stanley / k-Schur

April 12: Jianping on http://arxiv.org/abs/2103.09551

Apr 19: Erik.

Apr 26: Josh Petrack + Emi Brawley
Combinatorics of k-Schurs.