Recap: f = k-bounded affine permutation

best for geometry

- r_{ij} = cyclic rank matrix
- (u, w) w = k-Grassmannian permutation, $u \le w$ in Bruhat order

Fact: $u \le w$ iff $\quad u, w \in S_n$

$u(a) \le w(a)$ for $1 \le a \le k$

and

$u(b) \ge w(b)$ for $n \ge b > k$

provided that w is k-Grassmannian

$w(1) < \dots < w(k)$, $w(k+1) < \dots < w(n)$

Def $Gr(k,n) = \{ L \subset \mathbb{C}^n, \dim L = k \}$ = Grassmannian

$L = \text{Rowspan}\begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix} = \text{Rowspan}(V)$

v_i = columns of V

$V = k \times n$ matrix $\quad \text{rank}(V) = k$.

V is defined up to row operations.

Repeat v_i periodically: $v_1 \dots v_n, v_1 \dots v_n, \dots$

Def $r_{ij}(V) = \text{rank}(v_i, v_{i+1}, \dots, v_j)$ $\quad i \le j$

Key Lemma $r_{ij}(V)$ is a cyclic rank matrix (actually, $k - r_{ij}$ is a cyclic rank matrix in notations of last lecture).

Proof Need to check properties:

- $r_{ii} = r_{i+n, i+n}$

Note: we check properties: (... of last lecture)

- $r_{ij} = r_{i+n, j+n}$
- $r_{ij} = \underset{\text{rank}(V)}{k}$ for $j \geq i+n$ (used all the vectors in $v_i, \dots v_j$)
- $r_{i,j+1} = r_{ij}$ or $r_{ij}+1$ if $v_{j+1} \in \mathrm{Span}(v_i, \dots, v_j)$
- $r_{i+1,j} = r_{ij}$ or $r_{ij}-1$ if $v_i \in \mathrm{Span}(v_{i+1}, \dots v_j)$
- If $r_{i+1,j} = r_{i+1,j+1}$ and $r_{ij} = r_{i+1,j}$ then $r_{i+1,j} = r_{i,j+1}$

$v_i \in \mathrm{Span}(v_{i+1} \dots v_j)$

$v_{j+1} \in \mathrm{Span}(v_{i+1} \dots v_j)$

$$(v_i \ \underbrace{v_{i+1} \cdots v_j} \ v_{j+1})$$

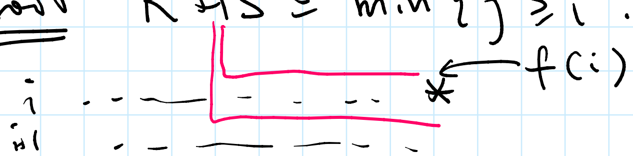
$$\mathrm{Span}(v_i \dots v_{j+1}) = \mathrm{Span}(v_{i+1}, \dots v_j)$$

Def An (open) positroid stratum $\mathring{\Pi}_f$ in $Gr(k,n)$ = subset of $Gr(k,n)$ with fixed $r_{ij} = r_{ij}(V)$

Rmk r_{ij} do not change under row operations $\Rightarrow$ $\mathring{\Pi}_f$ is a well defined subset of $Gr(k,n)$.

Lemma $f(i) = \min\{ j \geq i : v_i \in \mathrm{Span}(v_{i+1}, \dots v_j) \}$

Proof RHS $= \min\{ j \geq i : r_{ij} = r_{i+1,j} \}$



r_{ij} counts stars in

$\Rightarrow r_{ij} \neq r_{i+1,j}$ if $j < f(i)$ and $r_{ij} = r_{i+1,j}$ if $j \geq f(i)$

$\Rightarrow$ we reconstruct f from r_{ij} by the same rule. ■

Recall (r_{ij}) is bijection with f

3) we reconstruct f from r_{ij} by the same rule.

Recall (r_{ij}) is bijection with f.

Relation to Schubert cells in Gr

$$V \xleftrightarrow[\text{operations}]{\text{row}} \begin{array}{c} \begin{pmatrix} 0 \cdots 0 & 1 & * & 0 & * & 0 & * & 0 & * \\ 0 \cdots & & \cdots 0 & 1 & * & 0 & * & \vdots & * \\ 0 \cdots & & & & \cdots 0 & 1 & & & \\ 0 \cdots & & & & & & \cdots 0 & 1 & \end{pmatrix} \\ \quad\ \ \underbrace{\qquad i_1 \qquad\quad i_2}\quad\ i_3 \ \cdots\ i_k \end{array} \quad \mathrm{rref}(V)$$

Schubert cell in Gr $= \{V : \{i_1, \dots i_k\} = I\}$.
X_I

$= \{\dim \mathrm{Span}(v_1 \dots v_j) = \# I \cap \{1, \dots j\}\}$ for all j

how many pivot columns are there in the first i columns

$= \{r_{1j} = \# I \cap \{1 \dots j\}\}$ for all j

note: $j \in I$ iff $r_{1j-1} \neq r_{1j}$

So: $r_{1j} \longleftrightarrow$ subset I_1 and a Schubert cell X_{I_1}

$r_{2j} \longleftrightarrow$ subset I_2 and X_{I_2}

$\vdots$

$r_{nj} \longleftrightarrow I_n, X_{I_n}$

Cyclic rotation $\chi(v_1 \dots v_n) = (v_2, \dots v_n, v_1) \circlearrowright$ Gr
cyclic rotation

Thm $\mathring{\Pi}_f = X_{I_1} \cap \chi(X_{I_2}) \cap \chi^2(X_{I_3}) \dots \cap \chi^{n-1}(X_{I_n})$

where I_s are determined by r_{sj}.

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Ex $f(i) = i+k \Leftrightarrow$ Maximal positroid stratum $\mathring{\Pi}_{k,n} \subset Gr(k,n)$.

Exercise

$r_{i,i+k-1} = \text{rank}(v_i, \dots v_{i+k-1}) = k$

equivalently, $\Delta_{i,\dots,i+k-1} = \det(v_i, \dots v_{i+k-1}) \neq 0$

open subset of $Gr(k,n)$

Ex Even more concretely, $k=2$, $n=4$

$$\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix} \qquad \Delta_{ij} = \det\begin{pmatrix} | & | \\ v_i & v_j \\ | & | \end{pmatrix}$$

Plücker relation: $\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} = 0$ $(*)$

$\neq 0$

$$x = \frac{\Delta_{13}}{\Delta_{12}\Delta_{34}} \qquad y = \Delta_{24}$$

Divide $(*)$ by $\Delta_{12}\Delta_{34}$

$$1 - xy = -\frac{\Delta_{14}\Delta_{23}}{\Delta_{12}\,\Delta_{34}} \neq 0$$

Choose $\Delta_{12}, \Delta_{34}, \Delta_{23}$ arbitrarily $\neq 0$

x, y such that $1 - xy \neq 0$

$\Rightarrow \Delta_{14}$ is determined by these.

$$\mathring{\Pi}_{2,4} = \underbrace{\{xy \neq 1\}}_{\subset \mathbb{C}^2 \ \parallel \ \mathbb{C}^2 - \{\text{hyperbola}\}} \times (\mathbb{C}^*)^3$$

$\mathbb{C}^2 - \{\text{hyperbola}\}$.

$\mathbb{C}^2 - \{\text{hyperbola}\}$.

<u>Thm</u> (Galashin - Lam) If $GCD(k,n)=1$

(a) Point count in $\mathring{\Pi}_{k,n}$

$$\approx \frac{[n-1]_q!}{[k]_q!\,[n-k]_q!}(q-1)^?$$

(b) Homology of $\mathring{\Pi}_{k,n}$ with weight filtration $\approx$ (k,n) rational q,t-Catalan number.

<u>Ex</u> (a) $H^*(\mathbb{C}^2 - \{\text{hyperbola}\}) = ?$

(b) point count $(\mathbb{C}^2 - \{\text{hyperbola}\}) = ?$

For (b): $q^2 - \#\underset{q-1}{\underset{\|}{\{\text{hyperbola}\}}} = q^2-q+1 = \dfrac{q^3+1}{q+1}$

$\{\text{hyperbola}\} = \{xy=1\} \cong \{x \neq 0\}$

For (a): by Alexander duality,

$$\tilde{H}^*(\mathbb{C}^2 - \{\text{hyperbola}\}) = H_*(\text{hyperbola}) = H^*(\mathbb{C}^\times).$$

$\Rightarrow$ can compute that total dim of $H^*(\mathbb{C}^2 - \{\text{hyperbola}\}) = 3$

<u>Rmk</u> $J = (x_1 - x_2, y_1 - y_2) \subset \mathbb{C}[x_1, x_2, y_1, y_2]$

$KhR(T(2,2)) \to$ J/yJ = equivariant homology of

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Resolution of J: $a \leftarrow c \to b$ $\quad d(c) = (x_1 - x_2)a - (y_1 - y_2)b.$

$\langle a, b, c \rangle$ = non-equivariant homology of this space.

<u>Thm</u> (Galashin–Lam) Equivariant homology of $\mathring{\Pi}_{k,n} = KhR(T(k, n-k))$.

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④ Schubert cells in $Flag(n) = \{F_1 \subset F_2 \subset \dots \subset F_n\}$, $\dim F_i = i$

Schubert cells $\rightsquigarrow$ permutations

$\mathring{X}_w$ = Schubert cell for $w \in S_n$

$\mathring{X}^u$ = opposite Schubert cell for $u \in S_n$

$\pi : Flag(n) \longrightarrow Gr(k, n)$

$\{F_1 \dots F_n\} \longrightarrow F_k$

<u>Thm</u> [KLS] $\mathring{X}_w \cap \mathring{X}^u$ = Richardson variety in Flag

$$\pi(\mathring{X}_w \cap \mathring{X}^u) \overset{iso}{=} \mathring{\Pi}_f$$

where as above, w is k-grassmannian $u \leq w$ and $f \longleftrightarrow (w, u)$

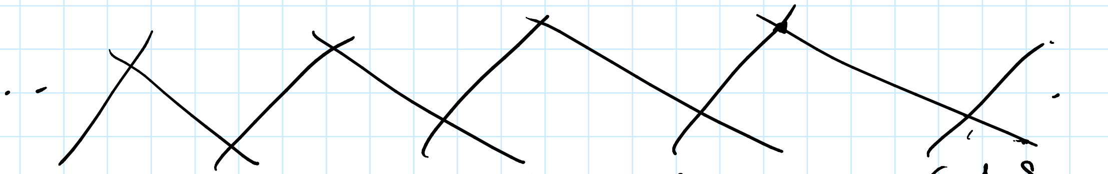
<u>Cor</u>: (a) $\mathring{\Pi}_f$ is smooth (because Richardson is)

(b) $\dim \mathring{\Pi}_f = \ell(w) - \ell(u)$

Ex $k=2, n=4$

Affine Springer fiber (ref. Erik's talks)

$$\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \qquad \{V \subset \mathbb{C}^2((t)) : tV \subset V,\ \gamma V \subset V\}$$



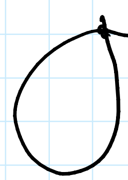
Spr

infinite chain of $(\mathbb{P}^1)$'s

What do we know?

- Has affine paving (1 pt, infinitely many lines)

  $\Rightarrow H^*$ is supported in even degrees
- Torus action $\mathbb{C}^* \circlearrowright$
- $\mathbb{Z}$ acts on this picture by translations (by 1)

Quotient $\mathrm{Spr}/\mathbb{Z} =$  $\mathbb{P}^1$ with 0 and ∞ glued

$$H^0 = H^1 = H^2 = \mathbb{Z}.$$

- $\mathrm{Spr}/\mathbb{Z}$ is a singular fiber in Hitchin fibration
- Def $H_*^{BM}(\mathrm{Spr}) \leftarrow$ module over $H_{T^*}(pt)$

  x, $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[y, y^{-1}]$.
- Goresky Kottwitz MacPherson

$\mathbb{C}[\mathbb{Z}] = \mathbb{C}[y, y^{-1}]$.

- Goresky-Kottwitz-MacPherson, Carlsson-Mellit, Kivinen... explicit $H^*(\mathrm{Spr})$.

$$\mathrm{Spr}\begin{pmatrix}1 & \\ & -1\end{pmatrix} \hookrightarrow \begin{pmatrix}t & 0\\ 0 & -t\end{pmatrix}$$

$$J \hookrightarrow \mathbb{C}[x,y]$$

$$KhR(T(n,n)) \hookrightarrow KhR(\mathrm{unlink})$$

$$\begin{pmatrix}t & 0\\ 0 & -t\end{pmatrix} = \gamma \qquad \det(\gamma - \lambda\cdot \mathrm{Id}) = (t-\lambda)(t+\lambda)$$

$$C = \det(\gamma - \lambda\cdot \mathrm{Id}) = 0$$