Some Combinatorics associated with k-Schur Functions

May 3, 2021

Def. ① An n-core is a shape where no n-ribbon can be removed.

Def. ② or... a shape where no cell has hook length equal to n.

Def. ③ or... a shape where no cell has hook length divisible by n.

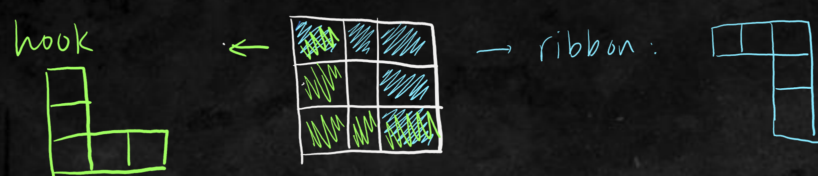
Note. I will use French notation for all our partitions, i.e.,

$(2,1) =$  .

These are all equivalent.

② $\Rightarrow$ ①: No hook length n $\Rightarrow$ no removable

② ⇒ ①: No hook length n ⇒ no removable n-ribbon. A removable n-ribbon has a northwest ↖ and southeast ↘ corner which are also the northwest and southeast corners of some hook. Both the ribbon and the hook are paths from the NW to the SE corner with steps going either east or south; so if the removable ribbon has length n, the corresponding hook must also have length n.



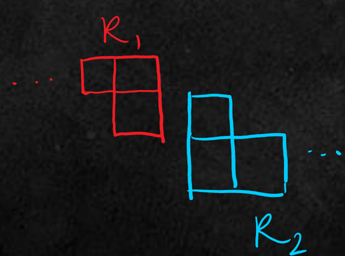
(Alternatively: push the hook up & right until it hits the edge of the diagram.)

③ ⇒ ②: No hook length divisible by n ⇒ no hook length equal to n.

Certainly $n \mid n$.

① ⇒ ③: No hook length n ⇒ no hook length divisible by n. Suppose K has a cell with hook length by n. Suppose κ has a cell with hook length kn, & hence a removable ribbon of length kn. Subdivide into $R_1, \dots, R_k$ (from NW to SE).

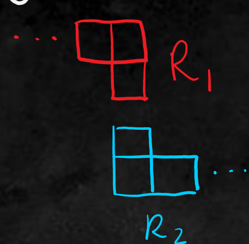
If R_1 is not removable:



(R_2 "blocking" R_1.)

exactly one happens

If R_1 "blocking" R_2:



If $R_1, \dots, R_{k-1}$ all non-removable, R_k must be blocking R_{k-1}, so R_{k-1} is <u>not</u> blocking R_k; so R_k is removable. ☺

removable kn-ribbon $\Rightarrow$ removable n-ribbon

Prop. There is a bijection

$$\{ n\text{-cores } K \mid |K|_n = m \} \longleftrightarrow \{ \lambda \vdash m \mid \lambda_1 \le n-1 \}$$

given by $\phi : K \mapsto \lambda$ with

$$\lambda_i = \# \{ (i,j) \in K \mid \mathrm{hook}_K(i,j) < n \}.$$

Note. $|K|_n$, also written $|K|$, is the number of cells in K with hook length $< n$.

Ex. $K =$ 

3-core

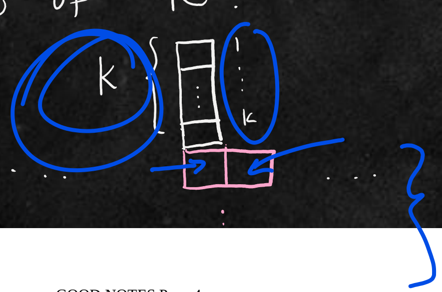
$\lambda_1 = \#\{\text{pink, blue}\} = 2$

$\lambda_2 = \#\{\text{yellow}\} = 1$

$\lambda = (2,1)$

Pf Prop. First: $\phi(K)$ is an $(n-1)$-bdd partition of m.

- λ_i counts cells in i^{th} column that contribute to $|K|$;
$$\sum \lambda_i = m.$$
- Hook length strictly increases as we go down;
$$\lambda_1 \le n-1.$$
- To show λ is a partition, look at i^{th}, $(i+1)^{st}$ columns of K.



Examine the first pair with

$$\ell < n,$$

$$\ell+k+1 \geq n+1. \quad (\Leftrightarrow \ell \geq n-k)$$

Below this pair, the $(i+1)^{st}$ column has at most hook lengths $n-k, n-k+1, \ldots, n-1$.

So

$$\lambda_{i+1} \leq \lambda_i.$$

i.e., λ is a partition.

Now, to show Φ is a bijection: build an inverse!

$$\underline{c} : \{ \lambda \vdash m \mid \lambda_1 \leq n-1 \} \longrightarrow \{ n\text{-cores } K \mid |K| = m \}$$

- Start at the top;
- If we have a cell with hook length n, slide right until we don't;

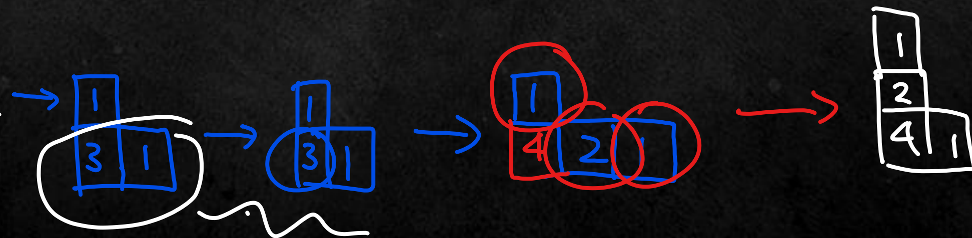
If we have a cell with ...

until we don't;

- Repeat, going down rows.

Conjugate.

Ex. $\lambda = (2,1)$.



Q: Does this really work (give us an n-core)?

Yes. :)

Sliding to the right decreases hook length of original cells, and newly inserted cells have increased hook length.

Note: $(n-1)$-boundedness is important!

So we have a bijection

$$\{n\text{-cores}\} \longleftrightarrow \{(n-1)\text{-bdd partitions}\}.$$